Hydrogen and galaxies (10 points)

This problem aims to study the peculiar physics of galaxies, such as their dynamics and structure. In particular, we explain how to measure the mass distribution of our galaxy from the inside. For this we will focus on hydrogen, its main constituent.

Throughout this problem we will only use $\hbar$, defined as $\hbar = h/2\pi$.

Part A - Introduction

Bohr model

We assume that the hydrogen atom consists of a non-relativistic electron, with mass m_e, orbiting a fixed proton. Throughout this part, we assume its motion is on a circular orbit.

A.1	Determine the electron's velocity v in a circular orbit of radius r.	0.2pt

SOLUTION:

Newton's second law on the electron in the electrical field of the proton for a circular orbit and projected on $\vec{u}_r$: $-m_e \frac{v^2}{r} = -\frac{e^2}{4\pi\varepsilon_0 r^2}$ hence $\boxed{v = \sqrt{\frac{e^2}{4\pi\varepsilon_0 m_e r}}}$

Marking Scheme

A.1.1 : Using Newton's second law	0.1
A.1.2 : Expression of the velocity	0.1

In the Bohr model, we assume the magnitude of the electron's angular momentum L is quantized, $L = n\hbar$ where $n > 0$ is an integer. We define $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx 7.27 \times 10^{-3}$.

A.2	Show that the radius of each orbit is given by $r_n = n^2 r_1$, where r_1 is called the Bohr radius. Express r_1 in terms of α, m_e, c and $\hbar$ and calculate its numerical value with 3 digits. Express v_1, the velocity on the orbit of radius r_1, in terms of α and c.	0.5pt

SOLUTION:

If the norm L of the angular momentum is quantified, for a circular orbit of radius r_n it is $L = m_e r_n v_n = n\hbar$. In the previous question, we have already obtained a relation between r and v that can be used for r_n and v_n and gives $v_n = \sqrt{\frac{e^2}{4\pi\varepsilon_0 m_e r_n}} = \sqrt{\frac{\alpha\hbar c}{m_e r_n}}$. Then using the quantifed expression we get $r_n = \frac{n\hbar}{m_e v_n} = \frac{n\hbar}{m_e}\sqrt{\frac{m_e r_n}{\alpha\hbar c}}$ thus $\boxed{r_n = \frac{\hbar n^2}{\alpha m_e c}}$ and then $\boxed{r_1 = \frac{\hbar}{\alpha m_e c}}$. For the numerical value we previously compute $\alpha = 7.27 \times 10^{-3}$ and then $\boxed{r_1 = 5.31 \times 10^{-11}\ \mathrm{m}}$. For the velocity, we get $m_e v_1^2 = \frac{e^2}{4\pi\varepsilon_0 r_1} = \frac{e^2 m_e v_1}{4\pi\varepsilon_0 \hbar}$ and then $\boxed{v_1 = \frac{e^2}{4\pi\varepsilon_0 \hbar} = \alpha c}$.

Marker Scheme

A.2.1 : Expression of r_n	0.1
A.2.2 : Expression of r_1	0.1
A.2.3 : Numerical value for r_1	0.1
A.2.4 : Expression of v_1	0.2

A.3	Determine the electron's mechanical energy E_n on an orbit of radius r_n in terms of e, ε_0, r_1 and n. Determine E_1 in the ground state in terms of α, m_e and c. Compute its numerical value in eV.	0.5pt

SOLUTION:

The mechanical energy is $E_n = \frac{1}{2}m_e v_n^2 - \frac{e^2}{4\pi\varepsilon_0 r_n} = -\frac{e^2}{8\pi\varepsilon_0 r_n}$, hence $\boxed{E_n = -\frac{e^2}{8\pi\varepsilon_0 n^2 r_1}}$ then for the ground state $\boxed{E_1 = -\frac{e^2}{8\pi\varepsilon_0 r_1}}$. Using the expression of α , we get the beautiful formula $\boxed{E_1 = -\frac{1}{2}\alpha^2 m_e c^2}$. The numerical value is $E_1 = -2.17 \times 10^{-18}$ J which corresponds to $\boxed{E_1 = -13.6\,\mathrm{eV}}$.

Marker Scheme

A.3.1 : Expression for E_n	0.2
A.3.2 : Expression for E_1 with α	0.2
A.3.3 : Numerical value for E_1	0.1

Hydrogen fine and hyperfine structures

The rare spontaneous inversion of the electron's spin causes a photon to be emitted on average once per 10 million years per hydrogen atom. This emission serves as a hydrogen tracer in the universe and is thus fundamental in astrophysics. We will study the transition responsible for this emission in two steps.

First, consider the interaction between the electron spin and the relative motion of the electron and the proton. Working in the electron's frame of reference, the proton orbits the electron at a distance r_1. This produces a magnetic field $\vec{B}_1$.

A.4	Determine the magnitude B_1 of $\vec{B}_1$ at the position of the electron in terms of μ_0, e, α, c and r_1.	0.5pt

SOLUTION:

The period of the motion is : $\boxed{T = \frac{2\pi r_1}{v_1}}$.

The current i corresponding to the orbit of the proton is $i = \frac{e}{T}$ hence $i = \frac{e v_1}{2\pi r_1} = \boxed{\frac{e\alpha c}{2\pi r_1}}$.

The magnetic field created by a loop with current i and radius R is : $\boxed{B = \frac{\mu_0 i}{2R}}$, which here gives $\boxed{B_1 = \frac{\mu_0 e\alpha c}{4\pi r_1^2}}$.

Marker Scheme

A.4.1 : Expression for the period	0.1
A.4.2 : Expression for the current	0.2
A.4.3 : General expression for B	0.1
A.4.4 : Inject i into B	0.1

Second, the electron spin creates a magnetic moment $\vec{\mathscr{M}}_s$. Its magnitude is roughly $\mathscr{M}_s = \frac{e}{m_e}\hbar$. The *fine* (F) structure is related to the energy difference ΔE_F between an electron with a magnetic moment $\vec{\mathscr{M}}_s$ parallel to $\vec{B}_1$ and that of an electron with $\vec{\mathscr{M}}_s$ anti-parallel to $\vec{B}_1$. Similarly, the *hyperfine* (HF) structure is related to the energy difference ΔE_HF, due to the interaction between parallel and anti-parallel magnetic moments of the electron and the proton. It is known to be approximately $\Delta E_\mathrm{HF} \simeq 3.72 \frac{m_e}{m_p} \Delta E_\mathrm{F}$ where m_p is the proton mass.

A.5	Express ΔE_F as a function of α and E_1. Express the wavelength λ_HF of a photon emitted during a transition between the two states of the hyperfine structure and give its numerical value with two digits.	0.5pt

SOLUTION:

The potential energy corresponding to the interaction between the spin magnetic moment $\vec{\mathscr{M}}_s$ and the nuclear magnetic field : $\boxed{E_p = -\vec{\mathscr{M}}_s \cdot \vec{B}_1}$

The difference ΔE_F between the energy of two electrons with a spin parallel and antiparallel to $\vec{B}_1$ is then $\boxed{\Delta E_\mathrm{F} = 2\mathscr{M}_s B_1}$. Using previous expressions one finds: $\Delta E_\mathrm{F} = 2\frac{e}{m_e}\hbar B_1 = 2\frac{e}{m_e}\hbar \frac{\mu_0 e \alpha c}{4\pi r_1^2}$ which writes $\boxed{\Delta E_\mathrm{F} = -4\alpha^2 E_1}$ hence $\Delta E_\mathrm{HF} = -3.72\frac{m_e}{m_p}4\alpha^2 E_1$.

The wavelength of the photon corresponding to this transition is then $\frac{hc}{\lambda_\mathrm{HF}} = \Delta E_\mathrm{HF} = -3.72.\frac{m_e}{m_p}4\alpha^2 E_1$ hence $\boxed{\lambda_\mathrm{HF} = -\frac{hc}{3.72.\frac{m_e}{m_p}4\alpha^2 E_1}}$ whose value is $\boxed{\lambda_\mathrm{HF} = 21\ \mathrm{cm}}$.

Marker Sheme

A.5.1 : Expression for the potential energy	0.1
A.5.2 : Expression for ΔE_F	0.1
A.5.3 : Expression for ΔE_HF in term of α	0.1
A.5.4 : Expression for λ_HF	0.1
A.5.5 : Numerical value for λ_HF	0.1

Part B - Rotation curves of galaxies

Data

- Kiloparsec: $1\,\mathrm{kpc} = 3.09 \times 10^{19}\,\mathrm{m}$
- Solar mass : $1\,\mathrm{M}_\odot = 1.99 \times 10^{30}\,\mathrm{kg}$

We consider a spherical galaxy centered around a fixed point O. At any point P, let $\rho = \rho(P)$ be the volumetric mass density and $\varphi = \varphi(P)$ the associated gravitational potential (i.e. potential energy per unit mass). Both ρ and φ depend only on $r = \left\|\overrightarrow{OP}\right\|$. The motion of a mass m located at P, due to the field φ, is restricted to a plane containing O.

B.1	In the case of a circular orbit, determine the velocity v_c of an object on a circular orbit passing through P in terms of r and $\frac{d\varphi}{dr}$.	0.2pt

SOLUTION:

The force created by the potential is $\vec{F} = -\vec{\nabla}(m\varphi(r)) = -m\frac{d\varphi}{dr}\vec{u}_r$. Newton's second law for a circular orbit then gives $m\frac{v_c^2}{r} = m\frac{d\varphi}{dr}$ hence $\boxed{v_c = \sqrt{r\frac{d\varphi}{dr}}}$.

SOLUTION:

B.1.1 : Using Newton's second law	0.1
B.1.2 : Expression for the velocity.	0.1

Fig. 1(A) is a picture of the spiral galaxy NGC 6946 in the visible band (from the $0.8\,\mathrm{m}$ Schulman Telescope at the Mount Lemmon Sky Center in Arizona). The little ellipses in Fig. 1(B) show experimental measurements of v_c for this galaxy. The central region ($r < 1\,\mathrm{kpc}$) is named the bulge. In this region, the mass distribution is roughly homogeneous. The red curve is a prediction for v_c if the system were homogeneous in the bulge and keplerian ($\varphi(r) = -\beta/r$ with $\beta > 0$) outside it, i.e. considering that the total mass of the galaxy is concentrated in the bulge.

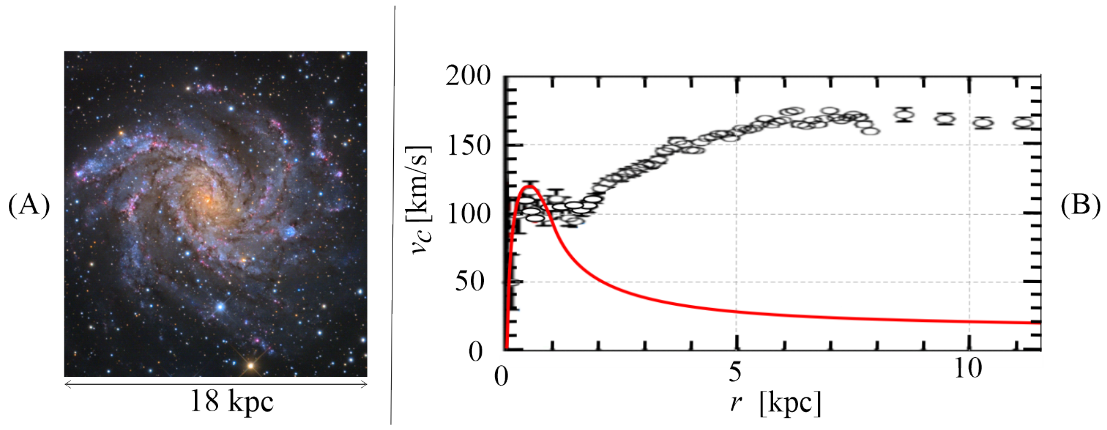


Fig. 1: NGC 6946 galaxy: Picture (A) and rotation curve (B).

B.2	Deduce the mass $M_b$of the bulge of NGC 6946 from the red rotation curve in Fig. 1(B), in solar mass units.	0.5pt

SOLUTION:

Either by Gauss's theorem $4\pi r^2 g(r) = -4\pi G M_{\mathrm{int}}(r)$, then one gets ~~$g(r) = GM_{\mathrm{int}}(r)/r^2$~~ $g(r) = -GM_{\mathrm{int}}(r)/r^2$. or one knows the law ~~$g(r) = GM/r^2$~~ $g(r) = -GM/r^2$ and intuits that one can use the interior mass

~~$g(r) = GM_{\text{int}}(r)/r^2$~~ $g(r) = -GM_{\text{int}}(r)/r^2$

If there is almost no more mass after the bulge radius r_b

then if $r > r_b$, $M_{\text{int}}(r) = M_b$ and $\boxed{\vec{g}(r > r_b) = -\frac{GM_b}{r^2}\vec{u}_r}$. But $\vec{g} = -\frac{d\varphi}{dr}\vec{u}_r$.

This gives $v_c(r > r_b) = \sqrt{\frac{GM_b}{r}}$.

One can then deduce that if the velocity is given only by the bulge, at a given distance R we must have $\boxed{M_b = v_c^2 R/G}$. On the red curve we can read $v_c = 20\ \text{km}\cdot\text{s}^{-1}$ at $R = 10$ kpc hence $M_b = \frac{v_c^2 R}{G} \simeq \frac{4.10^8 \times 3.10^{20}}{6{,}7.10^{-11}} \simeq 1{,}8.10^{39}$ kg so that $\boxed{M_b \simeq 9.10^8 M_\odot}$.

Marker Scheme

B.2.1 : ~~$g(r) = GM_{\text{int}}(r)/r^2$~~ $g(r) = -GM_{\text{int}}(r)/r^2$ via Gauss' Theorem or another method resulting in an equivalent result.	0.1
B.2.2 : Expression for $\vec{g}(r > r_b)$	0.1
B.2.3 : Expression for M_b	0.1
B.2.4 : Taking the right value of v_c in the figure	0.1
B.2.5 : Numerical value for M_b with a tolerance of $\pm 25\%$	0.1

Comparing the keplerian model and the experimental data makes astronomers confident that part of the mass is invisible in the picture. They thus suppose that the galaxy's actual mass density is given by

$$\rho_m(r) = \frac{C_m}{r_m^2 + r^2} \tag{1}$$

where $C_m > 0$ and $r_m > 0$ are constants.

B.3	Show that the velocity profile $v_{c,m}(r)$, corresponding to the mass density in Eq. 1, can be written $v_{c,m}(r) = \sqrt{k_1 - \frac{k_2 \cdot \arctan\left(\frac{r}{r_m}\right)}{r}}$. Express k_1 and k_2 in terms of C_m, r_m and G. (Hints: $\int_0^r \frac{x^2}{a^2 + x^2} dx = r - a \arctan(r/a)$, and: $\arctan(x) \simeq x - x^3/3$ for $x \ll 1$.) Simplify $v_{c,m}(r)$ when $r \ll r_m$ and when $r \gg r_m$. Show that if $r \gg r_m$, the mass $M_m(r)$ embedded in a sphere of radius r with the mass density given by Eq. 1 simplifies and depends only on C_m and r. Estimate the mass of the galaxy NGC 6946 actually present in the picture in Fig. 1(A).	1.8pt

SOLUTION:

On the one hand, writing Gauss' theorem on a sphere of radius r gives $\int \vec{g}(r) \cdot \vec{dS} = 4\pi r^2 g(r) = -4\pi GM_{\text{int}}$ and thus ~~$g(r) = GM_{\text{int}}(r)/r^2$~~ $g(r) = -GM_{\text{int}}(r)/r^2$. As long as this final formula is given it doesn't matter the method.

But, on the other hand $M_{\text{int}} = \int_0^r 4\pi x^2 \rho(x) dx = \boxed{4\pi C_m \left[r - r_m \arctan\left(\frac{r}{r_m}\right)\right]}$ hence

$$g_m(r) = -\frac{4\pi C_m G\left[r - r_m \arctan\left(\frac{r}{r_m}\right)\right]}{r^2} \tag{2}$$

But as $-m\frac{v_{c,m}^2}{r} = -mg_m(r)$ $-m\frac{v_{c,m}^2}{r} = mg_m(r)$ we finally get $v_{c,m} = \sqrt{rg_m(r)}$ $v_{c,m} = \sqrt{-rg_m(r)}$ wich writes

$$v_{c,m} = \sqrt{\frac{4\pi C_m G\left[r - r_m \arctan\left(\frac{r}{r_m}\right)\right]}{r}} \tag{3}$$

One can then read $k_1 = 4\pi C_m G$ and $k_2 = 4\pi C_m G r_m$

Two regime could be considered:

- if $r \ll r_m$, a third order Taylor expansion of arctan gives $v_{c,m} \simeq \sqrt{\frac{4\pi C_m G r^2}{3r_m^2}}$,
- and if $r \gg r_m$ then $\arctan\left(\frac{r}{r_m}\right) \simeq \pi/2$ and $v_{c,m} \simeq \sqrt{4\pi C_m G}$.

The function $v_{c,m}(r)$ is vanishing when $r \to 0$ and is asymptotically constant with value $\sqrt{4\pi C_m G}$ when $r \to +\infty$: this corresponds to the observational curve for the galaxy considered (black circles on the right part of figure 1(B). A natural interpretation for r_m is the typical radius beyond which the circular velocity is constant. On this picture one can read $v_c \simeq 160\,\mathrm{km \cdot s^{-1}}$ for the constant value of $v_{c,m}$ after r_m, then one can deduce $C_m = \frac{v_c^2}{4\pi G} \simeq \frac{(1{,}6.10^5)^2}{4\pi \times 6.67.10^{-11}} \simeq 3.10^{19}\,\mathrm{kg \cdot m^{-1}}$. The mass embedded in a sphere of radius r is given by $M_\mathrm{int} = \int_0^r 4\pi x^2 \rho_m(x)dx = 4\pi C_m\left[r - r_m \arctan\left(\frac{r}{r_m}\right)\right]$ which reduces to $M_\mathrm{int} \simeq 4\pi C_m r$ if $r \gg r_m$. In the picture we have a radius $R = 9\,\mathrm{kpc} = 2.27 \times 10^{20}\,\mathrm{m}$ of the galaxy, then a mass $M_\text{in the figure} \simeq 4\pi C_m R \simeq 10^{41}\,\mathrm{kg} \simeq 10^{11}\,\mathrm{M_\odot}$.
This mass corresponds to more than ten times the value of the mass actually visible in this picture : this is the dark matter concept.

Marker Scheme

B.3.1 : $g(r) = GM_\mathrm{int}(r)/r^2$ via Gauss' Theorem or another method resulting in an equivalent result.	0.2
B.3.2 : Interior mass	0.3
B.3.3. : Expression for $g(r)$	0.1
B.3.4 : Using Newton's second law	0.1
B.3.5 : Expression for k_1	0.1
B.3.6 : Expression for k_2	0.1
B.3.7 : Simplification for v_c in the case $r \ll r_m$	0.2
B.3.8 : Simplification for v_c in the case $r \gg r_m$	0.2
B.3.9 : Value of C_m	0.2
B.3.10 : Expression for M_m in the case $r \gg r_m$	0.2
B.3.11 : Mass in the figure (good if nearest power of ten)	0.1

Part C - Mass distribution in our galaxy

For a spiral galaxy, the model for Eq. 1 is modified and one usually considers the gravitational potential is given by $\varphi_G(r,z) = \varphi_0 \ln\left(\frac{r}{r_0}\right) \exp\left[-\left(\frac{z}{z_0}\right)^2\right]$, where z is the distance to the galactic plane (defined by $z = 0$), and $r < r_0$ is now the axial radius and $\varphi_0 > 0$ a constant to be determined. r_0 and z_0 are constant values.

C.1	Find the equation of motion on z for the vertical motion of a point mass m in such a potential, assuming r is constant. Show that, if $r < r_0$, the galactic plane is a stable equilibrium state by giving the angular frequency ω_0 of small oscillations around it.	0.5pt

SOLUTION:

The equation of motion is given by Newton's second law $m\vec{a} = \vec{F} = -m\vec{\nabla}\varphi$, projected on $\vec{u}_z$, it gives $m\ddot{z} = -m\frac{\partial\varphi}{\partial z}$. Using the given potential we have $\boxed{\ddot{z} = \frac{2z}{z_0^2}\varphi_0 \ln\left(\frac{r}{r_0}\right)\exp\left[-\left(\frac{z}{z_0}\right)^2\right]}$. Near the galactic plane ($z = 0$) the exponential is equal to 1 and can be simplified to give $\boxed{\ddot{z} \simeq \frac{2z}{z_0^2}\varphi_0 \ln\left(\frac{r}{r_0}\right)}$. If $r < r_0$ the ln is negative and the equation of motion is of the form $\ddot{z} \simeq -\omega_0^2 z$ with $\boxed{\omega_0 = \sqrt{\frac{2\varphi_0}{z_0^2}\left|\ln\left(\frac{r}{r_0}\right)\right|}}$. This proves that z is oscillating around $z = 0$ and that the motion is stable.

Marker Scheme

C.1.1 : Newton's second law, or equivalent method	0.1
C.1.2 : Projection on the z axis	0.1
C.1.3 : Equation of motion	0.1
C.1.4 : Equation near the galactic plane	0.1
C.1.5 : Expression for ω_0	0.1

From here on, we set $z = 0$.

C.2	Identify the regime, either $r \gg r_m$ or $r \ll r_m$, in which the model of Eq. 1 recovers a potential of the form $\varphi_G(r,0)$ with a suitable definition of φ_0. Under this condition $v_c(r)$ no longer depends on r. Express it in terms of φ_0.	0.6pt

SOLUTION:

Using the density given by equation (1) in part B, we have obtained

$$g_m(r) = -\frac{4\pi C_m G\left[r - r_m \arctan\left(\frac{r}{r_m}\right)\right]}{r^2} \quad (4)$$

Hence, considering $\boxed{r \gg r_m}$, one can simplify this relation to $g_m(r) \simeq -\frac{4\pi C_m G}{r}$. The gravitational potential can be obtained by integration, we then have : $\boxed{\varphi(r) = +4\pi C_m G\ln(r) + \text{cst}}$. The constant can be found by correctly choosing the origin of the potential. This potential corresponds to: $\varphi_G(r, z=0) = \varphi_0 \ln\left(\frac{r}{r_0}\right)$ with $\boxed{\varphi_0 = +4\pi C_m G}$. In that case, the equation of motion in the galactic plane gives $-m\frac{v_c^2}{r} = -mg_m(r)$ which writes $v_c = \sqrt{r g_m(r)} = \sqrt{4\pi C_m G}$, so that $\boxed{v_c = \sqrt{\varphi_0}}$.

Marker Scheme

C2.1 : Condition for simplication $r \gg r_m$	0.1
C2.2 : Expression for $\varphi(r)$	0.2
C2.3 : Identification of φ_0	0.1
C2.4 : Newton's second law	0.1
C2.5 : Expression for v_c	0.1

Therefore, outside the bulge the velocity modulus v_c does not depend on the distance to the galactic center. We will use this fact, as astronomers do, to measure the galaxy's mass distribution from the inside.

All galactic objects considered here for astronomical observations, such as stars or nebulae, are primarily composed of hydrogen. Outside the bulge, we assume that they rotate on circular orbits around the galactic center C. S is the sun's position and E that of a given galactic object emitting in the hydrogen spectrum. In the galactic plane, we consider a line of sight SE corresponding to the orientation of an observation, on the unit vector $\hat{u}_v$ (see Fig. 2).

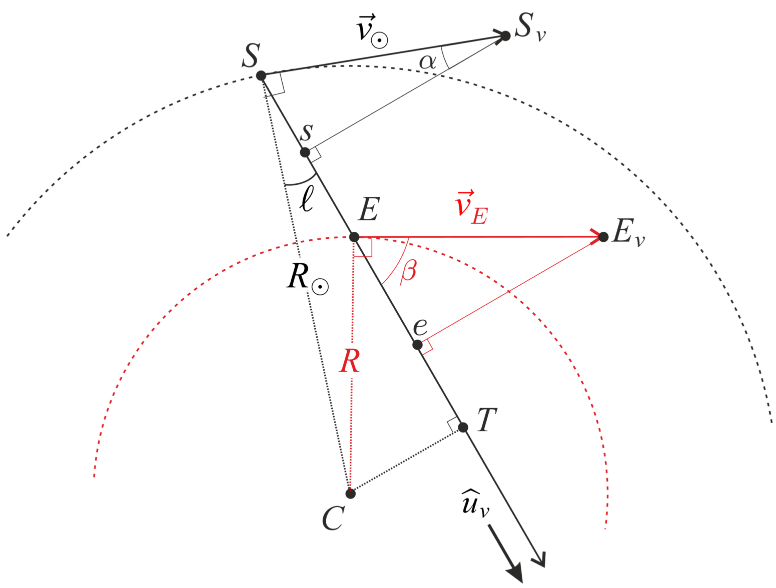


Fig. 2: Geometry of the measurement

Let ℓ be the galactic longitude, measuring the angle between SC and the SE. The sun's velocity on its circular orbit of radius $R_\odot = 8.00\,\text{kpc}$ is denoted $\vec{v}_\odot$. A galactic object in E orbits on another circle of radius R at velocity $\vec{v}_E$. Using a Doppler effect on the previously studied $21\,\text{cm}$ line, one can obtain the relative radial velocity $v_{rE/S}$ of the emitter E with respect to the sun S : it is the projection of $\vec{v}_E - \vec{v}_\odot$ on the line of sight.

C.3	Determine $v_{rE/S}$ in terms of ℓ, R, $R_\odot$ and $v_\odot$. Then, express R in terms of $R_\odot$, $v_\odot$, ℓ and $v_{rE/S}$.	0.7pt

SOLUTION:

We have $\boxed{\overrightarrow{Ss} = v_\odot \sin(\alpha)\hat{u}_v}$ and $\boxed{\overrightarrow{Ee} = v_E \cos(\beta)\hat{u}_v}$. In the right triangle SsS_v the sum of angles gives $\widehat{\left(\overrightarrow{Ss}, \overrightarrow{SS_v}\right)} = \frac{\pi}{2} - \alpha$, but, as $\vec{v}_\odot$ is perpendicular to the radius CS, we also have $\widehat{\left(\overrightarrow{Ss}, \overrightarrow{SS_v}\right)} = \frac{\pi}{2} - \ell$: then $\alpha = \ell$. On the other side, we have $CT = R_\odot \sin(\ell) = R\sin(\frac{\pi}{2} - \beta)$, which gives $\boxed{\cos(\beta) = \frac{R_\odot}{R}\sin(\ell)}$. Merging all of these results and taking into account that $v_E = v_\odot$ and that $\vec{v}_{rE/S} = \overrightarrow{Ee} - \overrightarrow{Ss}$ we have $\boxed{v_{rE/S} = v_\odot\left(\frac{R_\odot}{R} - 1\right)\sin(\ell)}$

and finally $\boxed{R = \dfrac{R_\odot}{1 + \frac{v_{rE/S}}{v_\odot \sin(\ell)}}}$.

Marker Scheme

C3.1 : Expression for $\overrightarrow{Ss}$	0.1
C3.2 : Expression for $\overrightarrow{Ee}$	0.1
C3.3 : $\alpha = \ell$	0.1
C3.4 : Expression for $\cos(\beta)$	0.1
C3.5 : Expression for $v_{r,E/S}$	0.2
C3.6 : Expression for R	0.1

Using a radio telescope, we make observations in the plane of our galaxy toward a longitude $\ell = 30°$. The frequency band used contains the 21 cm line, whose frequency is $f_0 = 1.42\,\text{GHz}$. The results are reported in Fig. 3.

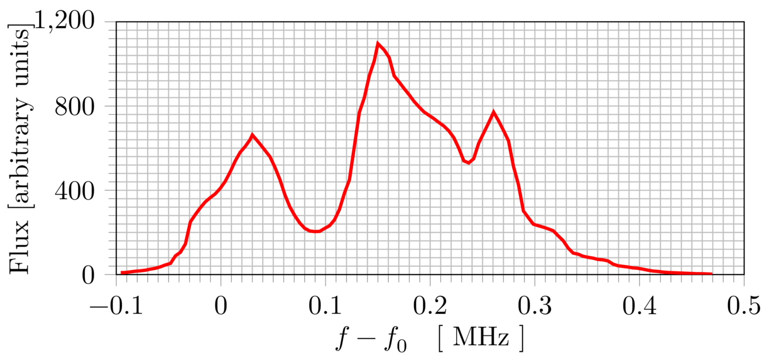


Fig. 3: Electromagnetic signal as a function of the frequency shift, measured in the radio frequency band at $\ell = 30°$ using EU-HOU RadioAstronomy

C.4	In our galaxy, $v_\odot = 220\,\text{km}\cdot\text{s}^{-1}$. Determine the values of the relative radial velocity (with 3 significant digits) and the distance from the galactic center (with 2 significant digits) of the 3 sources observed in Fig. 3. Distances should be expressed as multiples of $R_\odot$.	0.6pt

SOLUTION:

In Fig. 3 one can measure the 3 frequency shifts ($f - f_0$) corresponding to each peak : $\Delta f_1 = 0.03$ MHz , $\Delta f_2 = 0.15$ MHz and $f_3 = 0.26$ MHz. One can then compute the relative Doppler velocity using $\boxed{v_{r,i} = c\Delta f_i / f_0}$, with $f_0 = 1\,420$ MHz one gets

- $\boxed{v_{r,1} = 6.33\ \mathrm{km\cdot s^{-1}}}$
- $\boxed{v_{r,2} = 31.7\ \mathrm{km\cdot s^{-1}}}$
- $\boxed{v_{r,3} = 54.9\ \mathrm{km\cdot s^{-1}}}$

As peaks are placed on grid points, the tolerance in the value is due to fact that candidates could use $c = 3.00 \times 10^8$ m/s in the place of the 9 digits given in the formulary.

The corresponding distances from the galactic center are then obtained using the relation $R_i = \frac{R_\odot}{1 + \frac{v_{r,i}}{v_\odot \sin \ell}}$, with $\ell = 30°$ we obtain :

- $\boxed{R_1 = 0.95 R_\odot}$
- $\boxed{R_2 = 0.78 R_\odot}$
- $\boxed{R_3 = 0.67 R_\odot}$

Marker Scheme

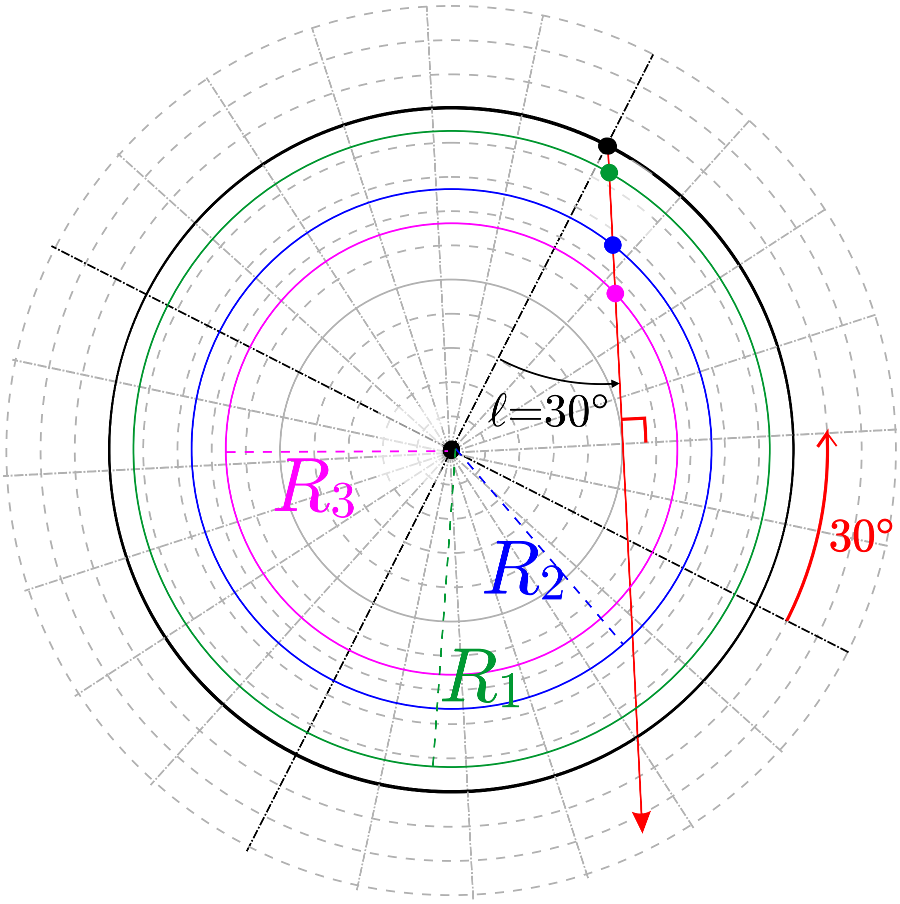
C4.1 : Doppler formula for v_r	0.1
C4.2 : Getting the 3 numerical values for Δ_f	0.2
C4.3 : Numerical values of the 3 velocities ($\pm 0.01\ \mathrm{km\cdot s^{-1}}$)	0.2
C4.4 : Numerical values of the 3 distances ($\pm 0.01\ R_\odot$)	0.1

C.5	On the top view of our galaxy (in the answer box), indicate the positions of the sources observed in Fig. 3. What could be deduced from repeated measurements changing ℓ?	0.6pt

SOLUTION:

As indicated on the figure below, the right line of sight could be obtained geometrically (i.e. without protractor) : using the 15° grid graduation one can going back from 30° from the perpendicular line to CS, we then obtain a radius which perpendicular to the line of sight, in other words as sin(30°)=0.5 the line of sight is passing by S and is tangenting the circle of radius CS/2.

Drawing the circles or radius , and , the line of sight with from we get 2 possible intersection for each peak : a near one and a far one. We plot only the nearest for each source on the answer figure.



The far intersections for each source is much further away and hence is likely less intense. Astronomers could also use the variation in the radio signal when they slowly vary the longitude to determine the right position of the actual source. A continuous variation of ℓ in the interval $[0, 2\pi]$ makes hydrogen sources appear in the galaxy, as the galaxy is essentially composed of hydrogen, one can trace its mass distribution : *i.e.* the spiral structure.

Marker Scheme

C5.1 : Getting the right line of sight	0.1
C5.2 : Drawing for the 3 circles	0.2
C5.3 : Drawing for the 3 points	0.2
C5.4 : Deduction	0.1

Part D - Tully-Fisher relation and MOND theory

The flat external velocity curve of NGC 6946 in Fig. 1 is a common property of spiral galaxies, as can be seen in Fig. 4 (left). Plotting the external constant velocity value $v_{c,\infty}$ as a function of the measured total mass M_{tot} of each galaxy gives an interesting correlation called the Tully-Fischer relation, see Fig. 4 (right).

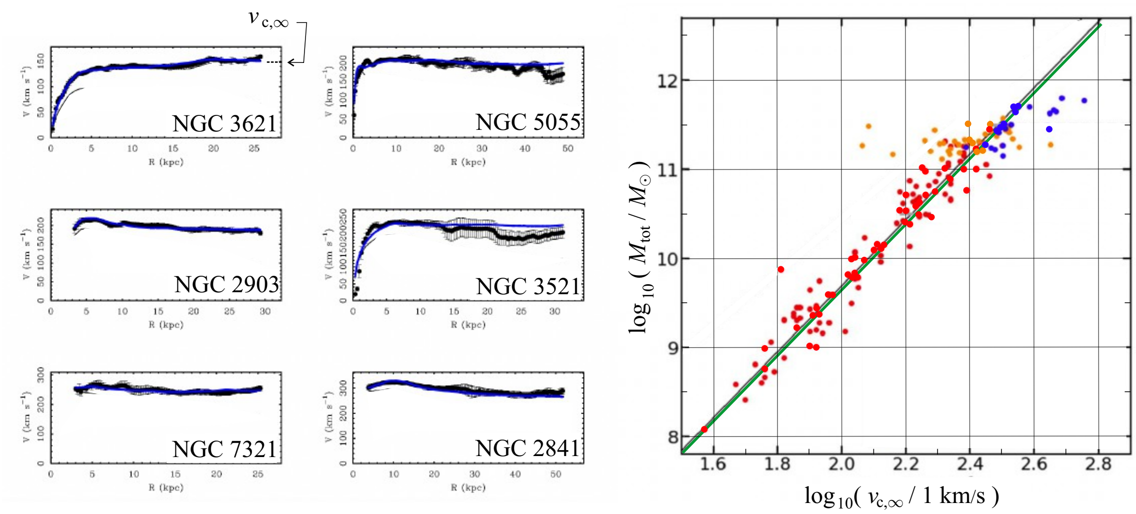


Fig. 4. Left: Rotation curves for typical spiral galaxies - Right: $\log_{10}(M_{\text{tot}})$ as a function of $\log_{10}(v_{c,\infty})$ on linear scales. Colored dots correspond to different galaxies and different surveys. The green line is the Tully-Fischer relation which is in very good agreement with the best fit line of the data (in black).

D.1	Assuming that the radius R of a galaxy doesn't depend on its mass, show that the model of Eq. 1 (part B) gives a relation of the form $M_{\text{tot}} = \eta v_{c,\infty}^{\gamma}$ where γ and η should be specified. Compare this expression to the Tully-Fischer relation by computing γ_{TF}.	0.4pt

SOLUTION:

We have obtained $v_{c,\infty}^2 = 4\pi C_m G$ and for a galaxy of radius R , we have $M_{\text{tot}} \simeq 4\pi C_m R$. This gives $C_m = \frac{M_{\text{tot}}}{4\pi R}$ and $v_{c,\infty}^2 = 4\pi \frac{M_{\text{tot}}}{4\pi R} G$. This relation is of the expected form $M_{\text{tot}} = \eta v_{c,\infty}^{\gamma}$ with $\boxed{\gamma = 2}$ and $\boxed{\eta = R/G}$. Analysing the data we get the power law exponent of the Tully-Fisher relation as $\boxed{\gamma_{TF} \simeq \frac{12-9}{2.6-1.8} = 3.75}$: the dark matter model from part B is not able to reproduce this law.

Marker Scheme

D1.1 : Recall for $v_{c,\infty}$	0.1
D1.2 : Expression for η	0.1
D1.3 : Expression for γ	0.1
D1.4 : Numerical value for γ_{TF} (correct if it is between 3.5 and 4)	0.1

In the extremely low acceleration regime, of the order of $a_0 = 10^{-10}\,\mathrm{m \cdot s^{-2}}$, the MOdified Newtonian Dynamics (MOND) theory suggests that one can modify Newton's second law using $\vec{F} = m\mu\left(\frac{a}{a_0}\right)\vec{a}$ where $a = \|\vec{a}\|$ is the modulus of the acceleration and the μ function is defined by $\mu(x) = \dfrac{x}{1+x}$.

D.2	Using data for NGC 6946 in Fig. 1, estimate, within Newton's theory, the modulus of the acceleration a_m of a mass in the outer regions of NGC 6946.	0.2pt

SOLUTION:

Considering that outer orbits are circular, the corresponding acceleration for a test mass m is radial and given in newtonian theory by $\boxed{a_m \simeq v_c^2/R}$. In the case of NGC 6946, the value of the velocity is roughly constant and equal to $v_c = 160\,\mathrm{km\cdot s^{-1}}$ as far $R > 5$ kpc. For this smallest distance from the center, the acceleration is $\boxed{a_m = \frac{(1.6.10^5)^2}{5.3.10^{19}} \simeq 1.5 \times 10^{-10}\,\mathrm{m\cdot s^{-2}}}$, this value is the maximal acceleration to which as star is submitted in the outer regions of this galaxy. It corresponds to the MOND regime.

Marker Scheme

D2.1 : Expression for a_m	0.1
D2.2 : Numerical value for a_m (good nearest power of ten)	0.1

D.3	Let m be a mass on a circular orbit of radius r with velocity $v_{c,\infty}$ in the gravity field of a fixed mass M. Within the MOND theory, with $a \ll a_0$, determine the Tully-Fischer exponent. Using data for NGC 6946 and/or Tully-Fischer law, calculate a_0 to show that MOND operates in the correct regime.	0.8pt

SOLUTION:

If $x = a/a_0 \ll 1$, then $\mu(x \ll 1) \simeq x$ and MOND theory gives $\vec{F} = m\frac{a}{a_0}\vec{a}$. Considering a gravitational interaction between M and m we then have for the radial component of the modified Newton's second Law $G\frac{M}{r^2}m = m\frac{a^2}{a_0}$. The radial acceleration on a circular orbit of radius r is always given by $a = v_{c,\infty}^2/r$, the modified second law writes now $G\frac{M}{r^2} = \frac{v_{c,\infty}^4}{r^2 a_0}$ which gives $\boxed{v_{c,\infty} = (a_0 GM)^{1/4}}$, and thus $M = \frac{1}{a_0 G}v_{c,\infty}^4$. Considering the notation from D.1, this is a power law relation with $\boxed{\gamma_{\mathrm{MOND}} = 4}$ in accordance with the Tully-Fischer relation.

For the NGC 6946 galaxy, we read $v_{c,\infty} = 160\,\mathrm{km\cdot s^{-1}}$ thus $\boxed{\log_{10}\left(\frac{v_{c,\infty}}{1\,\mathrm{km\cdot s^{-1}}}\right) = 2.2}$ and one can read the corresponding total mass by the Tully-Fischer relation as $\boxed{\log(M_{\mathrm{tot}}/M_\odot) = 10.5}$ thus $M_{\mathrm{tot}} = 2.10^{40,5}$ kg. One can obtain similar numbers using experimental data on the curve of Fig. 4. Introducing these values in the relation $\boxed{a_0 = \frac{v_{c,\infty}^4}{GM_{\mathrm{tot}}}}$ it gives $\boxed{a_0 = 1.5 \times 10^{-10}\,\mathrm{m\cdot s^{-2}}}$ as expected.

Marker Scheme

D3.1 : Considering the hypothesis $a \ll a_0$	0.1
D3.2 : Newton's second law	0.1
D3.3 : Expression for $v_{c,\infty}$	0.1
D3.4 : Numerical value for γ_{MOND}	0.1
D3.5 : Numerical value for $\log_{10}(v_{c,\infty}/1\,\mathrm{km/s})$	0.1
D3.6 : Numerical value for $\log_{10}(M)$	0.1
D3.7 : Expression for a_0	0.1
D3.8 : Numerical value for a_0 (good if nearest power of ten)	0.1

D.4 Considering relevant cases, determine $v_c(r)$ for all values of r in the MOND theory in the case of a gravitational field due to a homogeneously distributed mass M with radius R_b. 0.9pt

SOLUTION:

Taking the full formula for μ, the modified second law with circular velocity v_c at radius r writes now $\mathscr{G}(r)m = -m\frac{\frac{v_f^2}{a_0 r}}{1+\frac{v_f^2}{a_0 r}}\frac{v_f^2}{r}$ where $\mathscr{G}(r)$ is the gravitational field of the homogeneous ball of mass M and with radius R_b. This field can be deduced from Gauss' theorem it is

$$\mathscr{G}(r) = \begin{cases} \boxed{-GM/r^2} & \text{if} \quad r > R_b \\ \boxed{-GMr/R_b^3} & \text{if} \quad r \le R_b \end{cases} \tag{5}$$

Outside the ball : $r > R_b$. After a small reorganisation, v_c appears to be solution of the biquadratic equation $\boxed{v_c^4 - \frac{GM}{r}v_c^2 - a_0 GM = 0}$. The positive root of this equation is

$$\boxed{v_c(r) = \sqrt{\frac{GM}{2r}\left(1+\sqrt{1+\frac{4a_0 r^2}{GM}}\right)}} \quad \text{which is valid only if } r > R_b \tag{6}$$

When $r \to \infty$, v_c is asymptotically constant and $\boxed{M \to \frac{v_{c,\infty}^4}{a_0 G}}$ which is the Tully-Fisher relation. Inside the ball : $r \le R_b$. With a similar reorganisation, v_c appears now to be solution of another biquadratic equation which is $\boxed{v_c^4 - \frac{GM}{r}\left(\frac{r}{R_b}\right)^3 v_c^2 - a_0 GM\left(\frac{r}{R_b}\right)^3 = 0}$. The positive solution is now

$$\boxed{v_c(r) = \sqrt{\frac{GM}{2r}\left(\frac{r}{R_b}\right)^3\left[1+\sqrt{1+\frac{4a_0 r^2}{GM}\left(\frac{R_b}{r}\right)^3}\right]}} \quad \text{which is valid only if } r \le R_b \tag{7}$$

When $r \to 0$, we recover $\boxed{v_c \to 0}$ as in the experimental data.

Marker Scheme

D4.1 : Modified second law	0.1
D4.2 : Gravitational field in the case $r > R_b$	0.1
D4.3 : Gravitational field in the case $r < R_b$	0.1
D4.4 : Bi-quadratic equation in the case $r > R_b$	0.1
D4.5 : Expression for v_c in the case $r > R_b$	0.1
D4.6 : Behaviour in the limit $r \to \infty$	0.1
D4.7 : Bi-quadractic equation for $r < R_b$	0.1
D4.8 : Expression for v_c when $r < R_b$	0.1
D4.9 : Behaviour when $r \to 0$	0.1

Cox's Timepiece (10 points)

In 1765, British clockmaker James Cox invented a clock whose only source of energy is the fluctuations in atmospheric pressure. Cox's clock used two vessels containing mercury. Changes in atmospheric pressure caused mercury to move between the vessels, and the two vessels to move relative to each other. This movement acted as an energy source for the actual clock.

We propose an analysis of this device. Throughout, we assume that

- the Earth's gravitational field $\vec{g} = -g\,\vec{u_z}$ is uniform with $g = 9.8\,\mathrm{m\cdot s^{-2}}$ and $\vec{u_z}$ a unit vector;
- all liquids are incompressible and their density is denoted ρ;
- no surface tension effects will be considered;
- the variations of atmospheric pressure with altitude are neglected;
- the surrounding temperature T_a is uniform and all transformations are isothermal.

Fig. 1. Artistic view of Cox's clock [1]

Part A - Pulling on a submerged tube

We first consider a bath of water that occupies the semi-infinite space $z \le 0$. The air above it is at a pressure $P_a = P_0$. A cylindrical vertical tube of length $H = 1\,\mathrm{m}$, cross-sectional area $S = 10\,\mathrm{cm}^2$ and mass $m = 0.5\,\mathrm{kg}$ is dipped into the bath. The bottom end of the tube is open, and the top end of the tube is closed. We denote h the altitude of the top of the tube and z_ℓ that of the water inside the tube. The thickness of the tube walls is neglected.

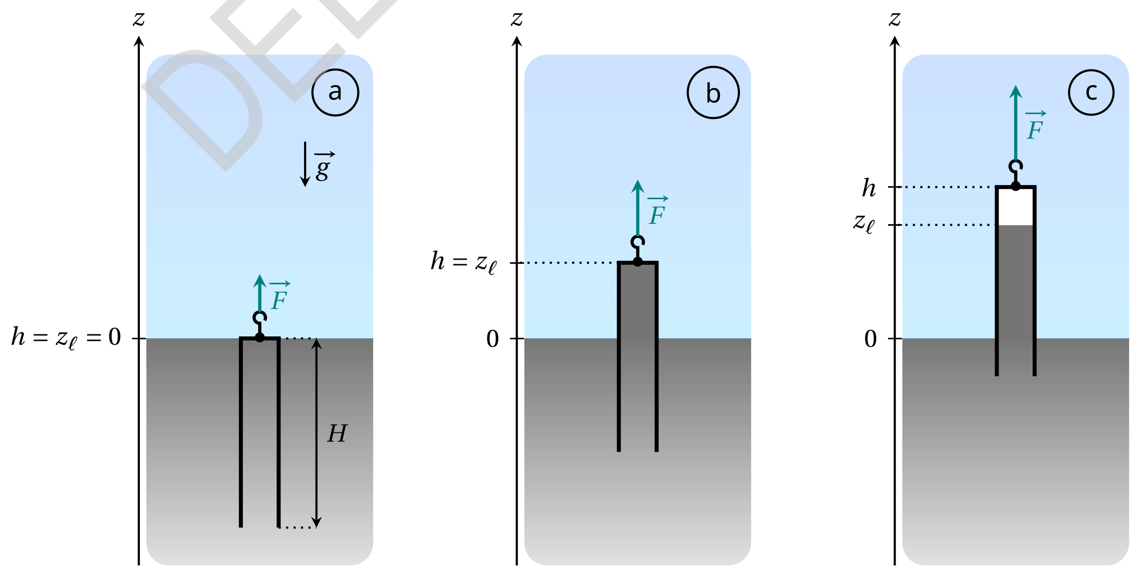


Fig. 2. Sketch of the tube in different configurations

We start from the situation where the tube in Fig. 2 contains no gas and its top is at the bath level: in other words, $h = 0$ and $z_\ell = 0$ (case a). The tube is then slowly lifted until its bottom end reaches the bath level. The pulling force exerted on the tube is denoted $\vec{F} = F\,\vec{u_z}$.

A.1	For the configuration shown in Fig. 2 (case b), express the pressure P_w in the water at the top of the tube. Also express the force $\vec{F}$ necessary to maintain the tube at this position. Expressions must be written in terms of P_0, ρ, m, S, h, g and $\vec{u_z}$.	0.2pt

SOLUTION:

According to the hydrostatic law, one has

$$P_\text{w} = P_\text{a} - \rho\, g\, h = P_0 - \rho\, g\, h$$

In the configuration shown in Fig. 2 (case b), the tube is submitted to three forces: its weight, the resultant of the pressure forces and the force exerted by the operator. Thus, at equilibrium, one has

$$\vec{0} = m\,\vec{g} + (P_\text{w} - P_0)\,S\,\vec{u_z} + \vec{F}$$

which leads to

$$\vec{F} = -\,[m + \rho\, S\, h]\,\vec{g} = [m + \rho\, S\, h]\, g\, \vec{u_z}$$

MARKING SCHEME:

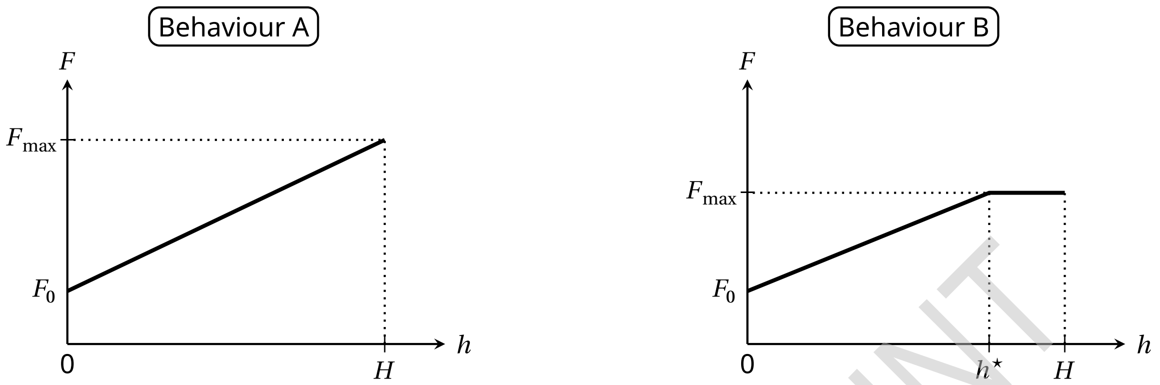
Expression of P_w (as a function of P_a or P_0)	0.1
Expression of $\vec{F}$	0.1

Three experiments are performed. In each, the tube is lifted from the initial state shown in Fig. 2(a) under the conditions specified in Table 1.

Experiment	Liquid	T_a (°C)	ρ (kg·m^{-3})	P_sat (Pa)
1	Water	20	1.00×10^3	2.34×10^3
2	Water	80	0.97×10^3	47.4×10^3
3	Water	99	0.96×10^3	99.8×10^3

Table 1. Experimental conditions and numerical values of physical quantities for each experiment (P_sat designates the saturated vapour pressure of the pure fluid)

In each case, we study the evolution of the force F that must be applied in order to maintain the tube in equilibrium at an altitude h, the external pressure being fixed at $P_\text{a} = P_0 = 1.000 \times 10^5\,\text{Pa}$. Two different behaviours are possible



A.2	For each experiment, complete the table in the answer sheet to indicate the expected behaviour and the numerical values for F_{max} and for $h^\star$ (when pertinent), where F_{max} and $h^\star$ are defined in the figures illustrating the two behaviours.	0.8pt

SOLUTION:

Physically, the altitude $h^\star$ corresponds to the threshold at which saturated vapour appears in the tube. This altitude can be expressed using the hydrostatic law, writing

$$P_w = P_0 - \rho g h^\star = P_{sat}(T_a).$$

One can find

$$h^\star = \frac{P_0 - P_{sat}(T_a)}{\rho g},$$

and calculate its numerical value for each experiment. If the value obtained is higher than H, behaviour A is observed; otherwise, behaviour B is observed. According to the previous question, the force F is related to h by

$$F = [m + \rho S h] g$$

which leads to

$$F_{max} = \begin{cases} [m + \rho S H] g & \text{for behaviour A} \\ [m + \rho S h^\star] g & \text{for behaviour B} \end{cases}$$

One can deduce the following predictions:

Experiment	Behaviour (A or B ?)	$h^\star$ (cm)	F_{max} (N)
1	A		14.7
2	A		14.4
3	B	2.1	5.1

MARKING SCHEME:

All behaviours are correct (*all or nothing*): A/A/B	0.2
Experiment 1: Numerical value of F_{max} in $[14.6, 15]$ (N)	0.1
Experiment 2: Numerical value of F_{max} in $[14, 14.5]$ (N)	0.1
Experiment 3: Numerical value of $h^\star$ in $[2, 2.2]$ (cm) (0.1 pt if only literal expression is correct)	0.2
Experiment 3: Numerical value of F_{max} in $[5, 5.2]$ (N) (0.1 pt if only literal expression is correct)	0.2

When we replace the water with liquid mercury (whose properties are given below), behaviour B is observed.

Liquid	T_a (°C)	ρ (kg·m^{-3})	P_{sat} (Pa)
Mercury	20	13.5×10^3	0.163

A.3	Express the relative error, denoted ε, committed when we evaluate the maximal force F_{max} neglecting P_{sat} compared to P_0. Give the numerical value of ε.	0.3pt

SOLUTION:

For behaviour B, the expression of F_{max} previously obtained can be reformulated as

$$F_{max} = m\,g + (P_0 - P_{sat})\,S$$

Neglecting the saturated vapour pressure compared to the atmospheric pressure, one obtains

$$F_{max} \simeq m\,g + P_0\,S$$

Thus, the relative error ε is given by

$$\varepsilon = \frac{P_{sat}}{P_0 + m\,g/S} \simeq 1.6 \times 10^{-6}$$

MARKING SCHEME:

Literal expression of ε (with or without P_{sat} in denominator)	0.2
Numerical value of ε in $[1, 2] \times 10^{-6}$	0.1

Part B - Two-part barometric tube

From now on, we work with mercury (density $\rho = 13.5 \times 10^3\,\mathrm{kg \cdot m^{-3}}$) at the ambient temperature $T_a = 20\,°\mathrm{C}$ and we take $P_{sat} = 0$.

Let us consider a tube with a reservoir on top, modeled as two superposed cylinders of different dimensions, as shown in Fig. 3.

- the bottom part (still called the tube) has cross-sectional area S_t and height $H_t = 80\,\mathrm{cm}$;
- the top part (called the bulb) has cross-sectional area $S_b > S_t$ and height $H_b = 20\,\mathrm{cm}$.

This two-part tube is dipped into a semi-infinite liquid bath.

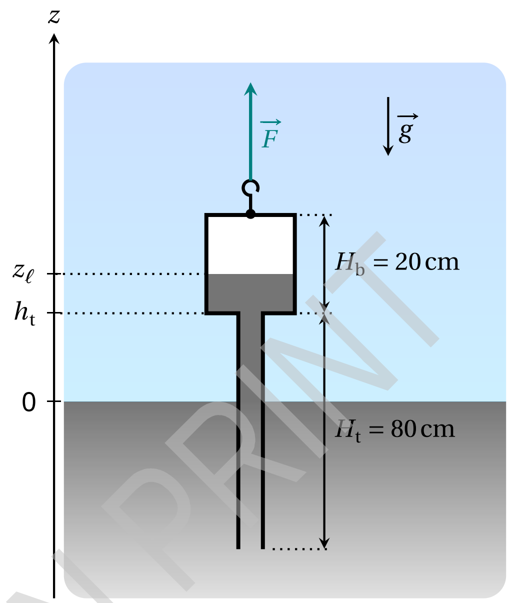


Fig. 3. Sketch of the two-part barometric tube

As in Part A, the system is prepared such that the tube contains no air. We identify the vertical position of the tube by the altitude h_t of the junction between the tube and the bulb. The height of the column of mercury is again denoted z_ℓ. The force $\vec{F}$ that must be exerted to maintain the tube in equilibrium in the configuration shown in Fig. 3 can now be written as

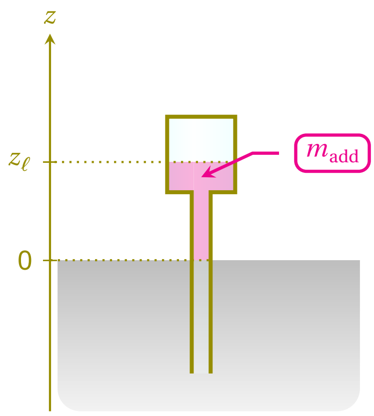
$$\vec{F} = (m_{tb} + m_{add})\, g\, \vec{u_z} \tag{1}$$

where m_{tb} is the total mass of the two-part tube (when empty of mercury).

B.1	On the answer sheet, color the area corresponding to the volume of liquid mercury that is responsible for the term m_{add} appearing in equation (1).	0.3pt

SOLUTION:

By adapting the reasoning used at part A, one can deduce that the mass m_{add} corresponds to the liquid mass in the two-part tube which is above the outside surface of the liquid bath, as shown below.



MARKING SCHEME:

Coloring of the correct area (0.1 pt only if a correct expression of m_{add} is provided but the colored area is incorrect)	0.3

The mass m_{add} depends both on the height h_t and the atmospheric pressure P_a. For the next question, assume that the atmospheric pressure is fixed at $P_a = P_0 = 1.000 \times 10^5\,\text{Pa}$. Starting from the situation where the system is completely submerged, the tube is slowly lifted until its base is flush with the liquid bath.

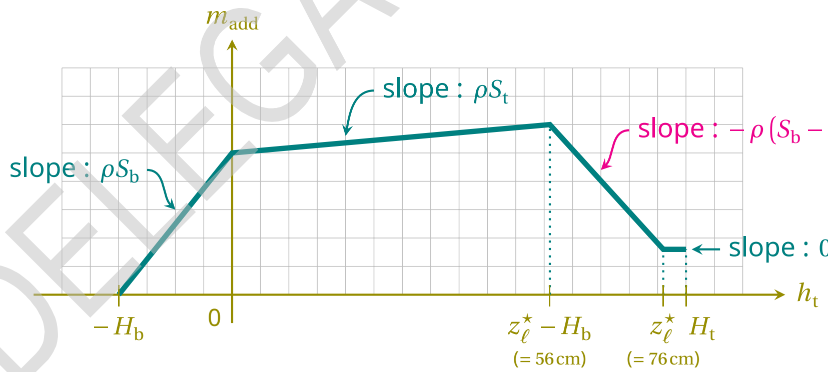
B.2	Sketch the evolution of the mass m_{add} as a function of h_t for $h_t \in [-H_b, H_t]$. On the graph, provide the expression for the slopes of the different segments, as well as the h_t analytical value of any angular points, in terms of P_0, ρ, g, S_b, S_t, H_b and H_t.	1.4pt

SOLUTION:

Using the same reasoning as in question A2, one can determine that saturated vapour appears in the two-part barometric tube when the altitude of the liquid column in the tube reaches the critical value

$$z_\ell^\star = \frac{P_0 - P_{\text{sat}}}{\rho g} = \frac{P_0}{\rho g} = 76\,\text{cm}$$

taking $P_{\text{sat}} = 0$. Combining this result with that of the previous question, one obtains the following graph:



MARKING SCHEME:

Qualitative aspect: Graph with 4 straight pieces (0.1pt only if there are 3 pieces; 0 else)	0.2
Qualitative aspect: For the 1st & 2nd pieces, the slopes are positive *and* the slope of 2nd piece is less than that of 1st (*all or nothing*)	0.2
Qualitative aspect: The 3rd piece has a negative slope	0.2
Qualitative aspect: The 4th piece has a null slope	0.2
Expressions of the two first slopes (*all or nothing*)	0.1
Expression of the negative slope	0.2
h_t analytical values of the 3 intermediate angular points (0.1pt per value)	0.3

As the system is lifted while $P_a = P_0 = 10^5\,\mathrm{Pa}$, we stop when the free surface of the liquid is in the middle of the bulb. The value of h_t is fixed and then we observe variations in the mass m_{add} due to variations in the atmospheric pressure described by

$$P_a(t) = P_0 + P_1(t) \tag{2}$$

where P_0 designates the average value and P_1 is a perturbative term. We model P_1 by a periodic triangular function of amplitude $A = 5 \times 10^2\,\mathrm{Pa}$ and period τ_1 of 1 week.

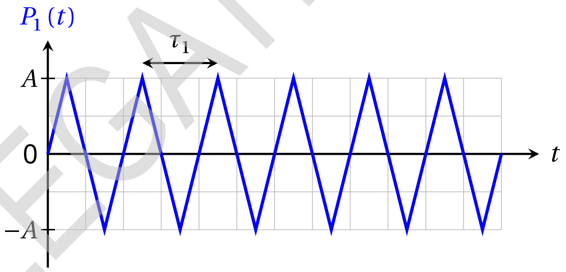


Fig. 4. Simplified model of the perturbative term $P_1(t)$

B.3	Given that $S_t = 5\,\mathrm{cm}^2$ and $S_b = 200\,\mathrm{cm}^2$, express the amplitude Δm_{add} of the variations of the mass m_{add} over time, then give its numerical value. *Assume that the liquid surface always stays in the bulb.*	0.3pt

SOLUTION:

By neglecting the saturated vapour pressure in the bulb, the altitude z_ℓ of the free surface of the liquid in the tube is given by

$$z_\ell(t) = \frac{P_a(t)}{\rho g} = \frac{P_0}{\rho g} + \frac{P_1(t)}{\rho g} = \underbrace{h_t + \frac{H_b}{2}}_{\text{mean value } z_{\ell,0}} + \underbrace{\frac{P_1(t)}{\rho g}}_{\text{perturbative term}}$$

which leads to

$$m_{\text{add}}(t) = \rho[S_t h_t + S_b(z_\ell(t) - h_t)] = \rho\left[S_t h_t + S_b(z_{\ell,0} - h_t)\right] + \frac{S_b P_1(t)}{g}$$

The first term gives the mean value of the mass $m_{\text{add}}(t)$, while the last term characterizes its temporal variations. One can deduce the magnitude

$$\Delta m_{\text{add}} = \frac{S_b A}{g} \simeq 1\,\text{kg}$$

MARKING SCHEME:

Literal expression of Δm_{add}	0.2
Numerical value *with unit*, in $[1\,\text{kg}, 1.1\,\text{kg}]$	0.1

Part C - Cox's timepiece

The real mechanism developed by Cox is complex (Fig. 5). We study a simplified version, depicted in Fig. 6, and described below

- a cylindrical bottom cistern containing a mercury bath ;
- a two-part barometric tube identical to that studied in part B, which is still completely emptied of any air, is dipped into the bath ;
- the cistern and the two-part tube are each suspended by a cable. Both cables (assumed to be inextensible and of negligible mass) pass through a system of ideal pullies and finish attached to either side of the same mass M, which can slide on a horizontal surface ;
- the total volume of liquid mercury contained in the system is $V_\ell = 5\,\text{L}$.

The height, cross-section and masses of each part are given in Table 2. The position of mass M is referenced by the coordinate x of its center of mass. We consider solid friction between the horizontal support and the mass M, without distinction between static and dynamic coefficients; the magnitude of this force when sliding occurs is denoted F_s.

Two stops limit the displacement of the mass M such that $-X \le x \le X$ (with $X > 0$). Assume that the value of X guarantees that

- the bottom of the two-part tube never touches the bottom of the cistern nor comes out of the liquid bath;
- the altitude z_ℓ of the mercury column is always in the upper bulb.

Fig. 5. Real Cox's timepiece [2] (without mercury)

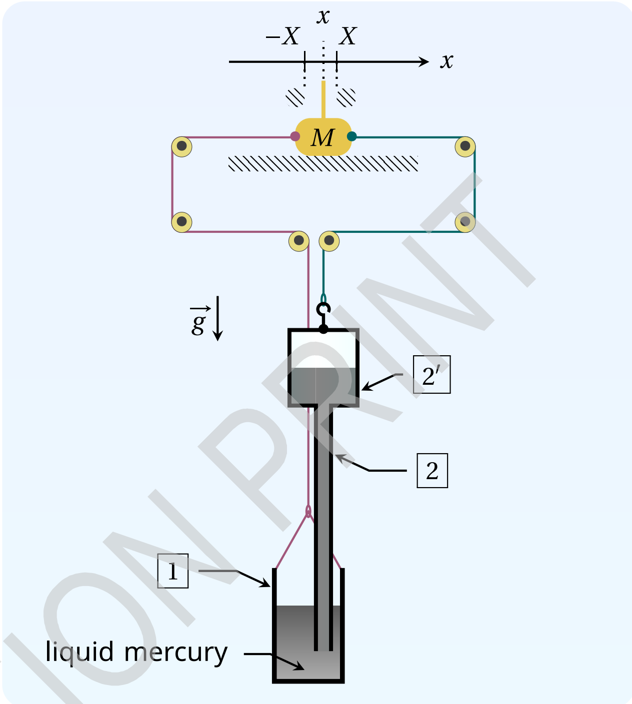


Fig. 6. Sketch of the system modeling the timepiece

Reference	Name	Height	Cross section area	Empty mass
1	cistern	$H_c = 30\,\text{cm}$	$S_c = 210\,\text{cm}^2$	m_c
2	tubular part of the barometric tube	$H_t = 80\,\text{cm}$	$S_t = 5\,\text{cm}^2$	total mass of the barometric tube : m_{tb}
2′	bulb of the barometric tube	$H_b = 20\,\text{cm}$	$S_b = 200\,\text{cm}^2$	

Table 2. Dimensions and notations for the model system

The system evolves in contact with the atmosphere, whose pressure fluctuates as in Fig. 4 (still with amplitude $A = 5 \times 10^2\,\text{Pa}$ and period $\tau_1 = 1\,\text{week}$). At the start $t = 0$, the mass M is at rest at $x = 0$ and the tensions exerted by the two cables on either side of the mass M are in balance while $P_1(0) = 0$. We define

$$\xi = \frac{S_b + S_c - S_t}{S_b S_c}\frac{F_s}{A} \simeq \frac{S_b + S_c}{S_b S_c}\frac{F_s}{A} \tag{3}$$

where the last expression uses that $S_t \ll S_b, S_c$ (which we will assume is valid until the end of the problem).

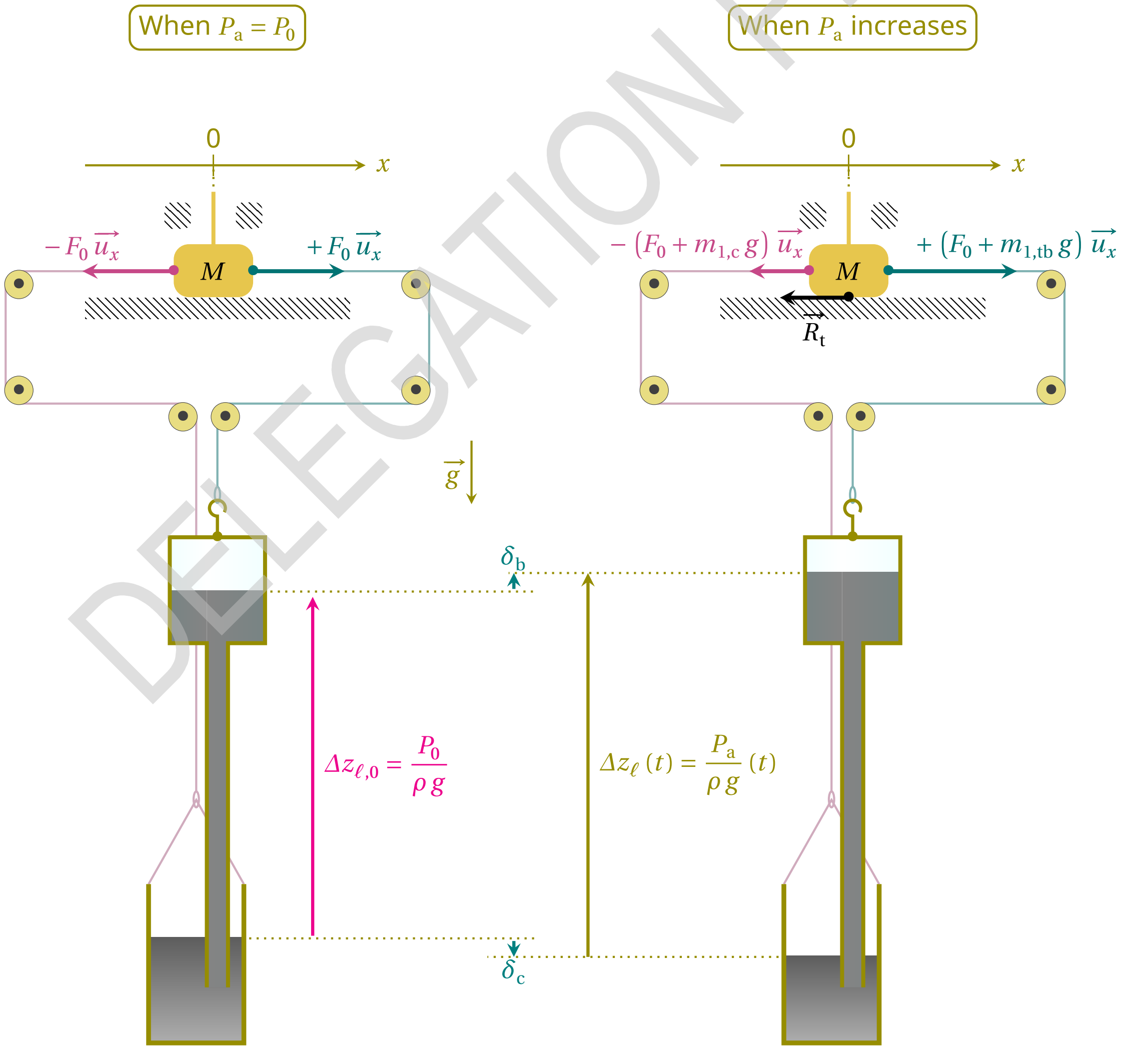
C.1	Determine the threshold $\xi^\star$ such that M remains indefinitely at rest when $\xi > \xi^\star$.	1pt

SOLUTION:

Consider the case in which the mass M stays at rest at $x = 0$. At the start $t = 0$, the tensions exerted by the two cables on either side of the mass M are in balance: the force F_0 required to suspend the barometric tube (with the fluid it contains) is equal to that required to suspend the cistern (with the fluid it contains). When the atmospheric pressure increases from $P_a = P_0$, the fluid rises in the barometric tube while it descends in the cistern. As a result, the added mass in the tube increases, while the added mass in the cistern decreases. We denote $m_{1,tb}$ and $m_{1,c}$ the (algebraic) variation of the apparent masses of each container. Thus, the tensions exerted by the two cables can be written:

- $\left[F_0 + m_{1,tb}\, g\right] \overrightarrow{u_x}$ for the cable on the right, suspending the tube;
- $-\left[F_0 + m_{1,c}\, g\right] \overrightarrow{u_x}$ for the cable on the left, suspending the cistern.

According to the principle of mass conservation, one can immediately state that $m_{1,tb} = -m_{1,c}$. Subsequently, we choose to keep only $m_{1,c}$ in the expressions (but all the calculations can be carried out while keeping $m_{1,tb}$).



The friction force between the support and the mass M needed to maintain the equilibrium is therefore given by

$$\vec{R_t} = -\left[F_0 - m_{1,c}\,g\right]\vec{u_x} + \left[F_0 + m_{1,c}\,g\right]\vec{u_x} = 2\,m_{1,c}\,g\,\vec{u_x}$$

In addition, according to the sketch above (where displacements δ_b and δ_c are algebraic), we have $m_{1,c} = \rho\,S_c\,\delta_c$.

It is now necessary to determine δ_c. One can use

- the hydrostatic law : $\delta_b - \delta_c = \frac{P_1}{\rho\,g}$
- the conservation of the total volume/mass of mercury : $S_b\,\delta_b = -\left[S_c - S_t\right]\delta_c \simeq -S_c\,\delta_c$ (given that $S_t \ll S_b, S_c$)

Solving the system formed by those equations, one finds

$$\delta_c = -\frac{S_b}{S_b + S_c - S_t}\frac{P_1}{\rho\,g} \simeq -\frac{S_b}{S_b + S_c}\frac{P_1}{\rho\,g}$$

which finally yields

$$\vec{R_t} = -\frac{2\,S_b\,S_c}{S_b + S_c - S_t}\,P_1\,\vec{u_x} \simeq -\frac{2\,S_b\,S_c}{S_b + S_c}\,P_1\,\vec{u_x}$$

With the triangular model for $P_1(t)$, the maximum static friction force is obtained when $P_1 = \pm A$. Therefore, according to the Coulomb's law of friction, the mass M stays at rest if and only if

$$\frac{2\,S_b\,S_c}{S_b + S_c - S_t}\,A < F_s$$

This inequality can be rewritten as

$$2 < \frac{S_b + S_c - S_t}{S_b\,S_c}\,\frac{F_s}{A} = \xi$$

which allows us to identify

$$\xi^\star = 2$$

MARKING SCHEME:

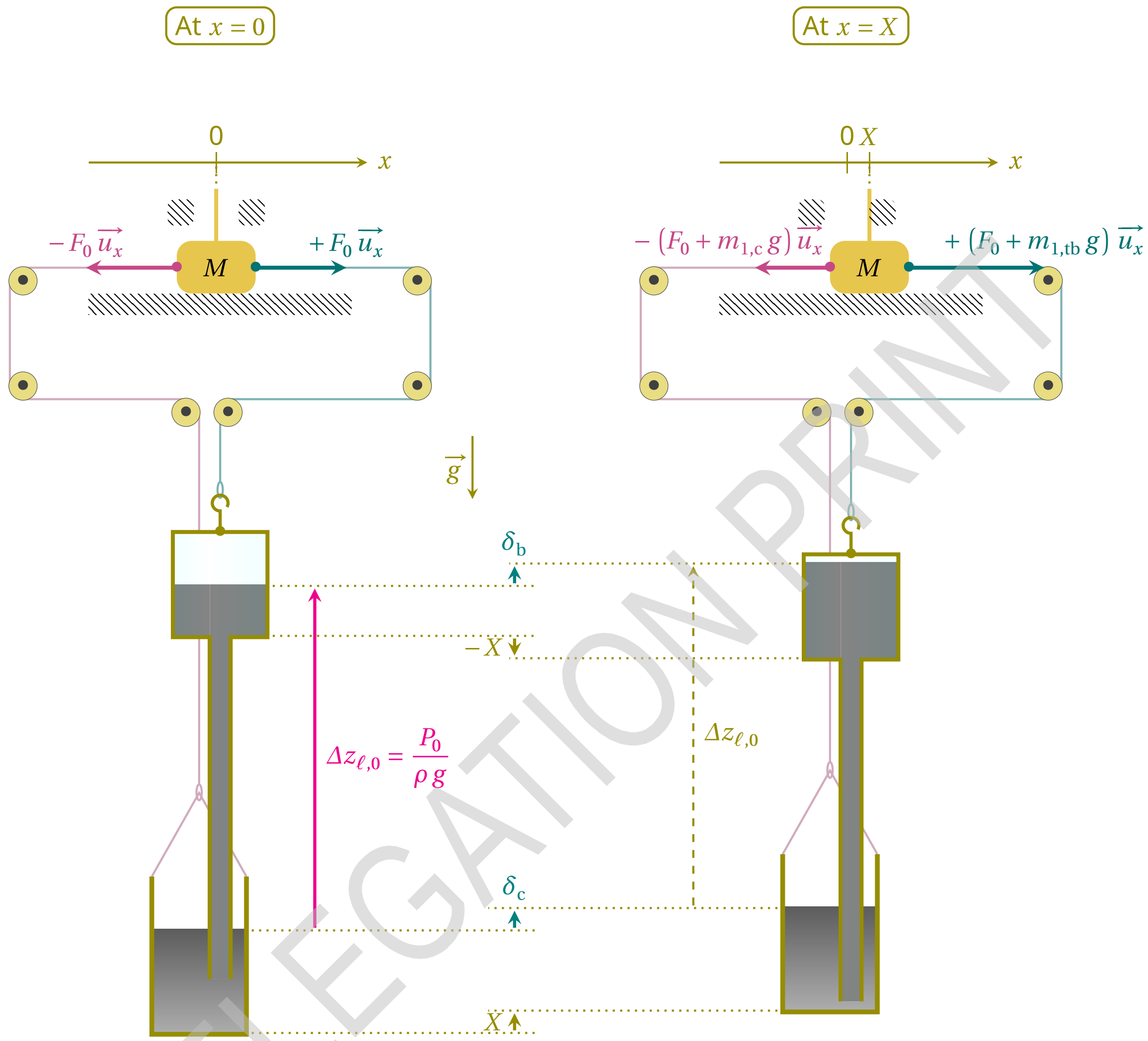
Introduction of geometric parameters to locate the positions of the fluid surfaces in each vessel	0.1
Expression of mass or volume variation of fluid in at least one of the vessels, in terms of those geometric parameters (with or without using $S_t \ll S_b, S_c$)	0.1
Physical law: Conservation of the total mass/volume	0.2
Physical law: Expression of barometric difference of heights between the two surfaces	0.2
Physical law: Expression of the friction force at equilibrium (with or without using $S_t \ll S_b, S_c$)	0.1
Physical law: Use of Coulomb's law in sticky situation	0.1
Conclusion: Obtaining ξ^*	0.2

For the next question only, suppose that the mass M is temporarily blocked at $x = X$.

C.2 Give an expression for the total tension force $\vec{T} = T\,\vec{u_x}$ acting on the mass M due to the tension in two cables at this position, when $P_1 = 0$, in terms of ρ, g, X and pertinent cross-sections. 1pt

SOLUTION:

Let us compare the configurations of the system when $x = 0$ and when $x = X$.



Assuming that the atmospheric pressure is temporarily fixed at P_0, the difference Δz_ℓ of fluid heights between the cistern and the barometric tube is the same in both configurations. It is given by $\Delta z_{\ell,0} = P_0 / \rho g$ and leads to

$$\delta_b = \delta_c$$

The total volume/mass of mercury is also conserved. This conservation can be expressed by the equation

$$\underbrace{(S_c - S_t)\,\delta_c - (S_c + S_t)\,X}_{\substack{\text{volume of mercury}\\ \text{algebraically won by the cistern}}} + \underbrace{S_b\,(\delta_b + X)}_{\substack{\text{volume of mercury}\\ \text{algebraically won by the bulb}}} = 0$$

which can be reformulated as

$$S_b\,\delta_b + (S_c - S_t)\,\delta_c = (S_c - S_b + S_t)\,X$$

One obtains

$$\delta_b = \delta_c = \frac{S_c - S_b + S_t}{S_b + S_c - S_t} X$$

Thus, the supplementary added mass in the cistern is given by

$$m_{1,c} = \rho\, S_c\,(\delta_c - X) = -\rho\,\frac{2S_c\,(S_b - S_t)}{S_c + S_b - S_t} X \simeq -\frac{2\,S_b\,S_c}{S_b + S_c}\,\rho\, X$$

and, as explained in C1, we still have $m_{1,tb} = -m_{1,c}$.

Finally, according to the sketch, one obtain the resultant tension force $\vec{T} = (m_{1,tb} - m_{1,c})\, g\, \vec{u_x} = -2\, m_{1,c}\, g\, \vec{u_x}$, that is

$$\vec{T} = \frac{4S_c\,(S_b - S_t)}{S_b + S_c - S_t}\,\rho\, g\, X\, \vec{u_x} \simeq \frac{4\,S_b\,S_c}{S_b + S_c}\,\rho\, g\, X\, \vec{u_x}$$

MARKING SCHEME:

Introduction of geometric parameters to locate the positions of the fluid surfaces in each vessel	0.1
Expressions of mass or volume variations of fluid in one of the vessels in terms of X and those geometric parameters (with or without using $S_t \ll S_b, S_c$)	0.3
Physical law: Conservation of the total mass/volume	0.2
Physical law: Expression of barometric difference of heights between the two surfaces	0.2
Expression of the total tension force $\vec{T}$ (with or without using $S_t \ll S_b, S_c$)	0.2

When $\xi < \xi^\star$, starting again from $x = 0$ and $P_1 = 0$, two different behaviours can be observed for $t \geq 0$. To distinguish them, we need to introduce another parameter

$$\lambda = \frac{2\,(S_b - S_t)}{S_b}\,\frac{\rho\, g\, X}{A} \simeq \frac{2\,\rho\, g\, X}{A} \tag{4}$$

C.3	Complete the table in the answer sheet to indicate the condition under which each regime is obtained. Conditions must be expressed as inequalities on ξ and/or λ. In addition, sketch the variations of $x\,(t)\,/X$ for $t \in [0,\, 3\,\tau_1]$ that are consistent with the variations of $P_1\,(t)\,/A$ already present. *Specification of remarkable points coordinates is not required.*	2pt

SOLUTION:

When $\xi < \xi^\star$, there necessarily exists an instant from which the mass M begins to sweep on the right. From there, the mass M is continuously accelerated by the total tension $\vec{T}$ until it is blocked by the stop at $x = X$. According to Fig. 5, one can assume that X is of the order of a few centimeters, so the time needed to switch between the two positions $x = 0$ and $x = X$ can reasonably be neglected in front of the period τ_1.

Once blocked at $x = X$, the resultant tension $\vec{T}$ can be determined by generalizing the reasoning carried out in the two previous questions. One obtains the following equations:

- hydrostatic law : $\delta_b - \delta_c = \dfrac{P_1}{\rho g}$
- conservation of the volume/mass : $S_b \delta_b + (S_c - S_t)\delta_c = (S_c - S_b + S_t) X$

The resolution of this system gives

$$\delta_c = \frac{S_c - S_b + S_t}{S_b + S_c - S_t} X - \frac{S_b}{S_b + S_c - S_t} \frac{P_1}{\rho g}$$

from which we deduce the perturbative added mass

$$m_{1,c} = \rho S_c (\delta_c - X) = -\rho \left[\frac{2 S_c (S_b - S_t)}{S_b + S_c - S_t} X + \frac{S_b S_c}{S_b + S_c - S_t} \frac{P_1}{\rho g} \right] = -m_{1,tb}$$

Then finally

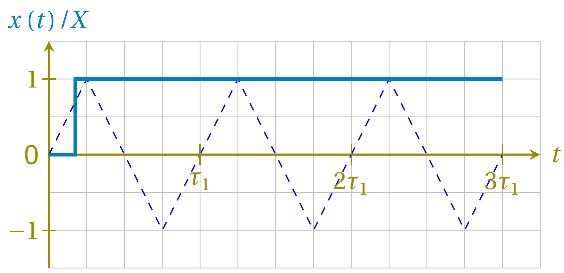
$$\vec{T} = -2 m_{1,c}\, g\, \vec{u_x} = \left[\frac{4 S_c (S_b - S_t)}{S_b + S_c - S_t} \rho g X + \frac{2 S_b S_c}{S_b + S_c - S_t} P_1 \right] \vec{u_x}$$

According to Coulomb's law of friction, the mass M will stay at rest at the position $x = X$ while $\vec{T} \cdot \vec{u_x} > -F_s$. With the model adopted for $P_1(t)$, this condition is always satisfied if

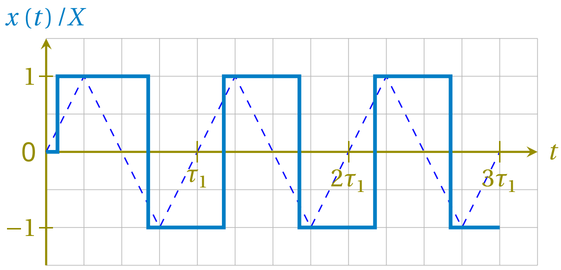
$$\frac{4 S_c (S_b - S_t)}{S_b + S_c - S_t} \rho g X - \frac{2 S_b S_c}{S_b + S_c - S_t} A > -F_s$$

Hence, using the parameters ξ and λ, one can identify the two possible regimes :

- Regime 1 : $\xi + 2\lambda > 2$ (once at $x = X$, the mass M stays indefinitely at rest)



- Regime 2 : $\xi + 2\lambda < 2$ (once at $x = X$, the mass M will periodically sweep between the two stops)



MARKING SCHEME:

Expression for $\vec{T}$ in the general case, containing both P_1 and X terms.	0.2
At least one inequality is correct (without considering strict or large)	0.2
Both inequalities are correct (without considering strict or large)	0.1
Global appearance of *both* graphs: one seems to show an aperiodic behaviour, the other a periodic behaviour (*all or nothing*)	0.2
Global appearance: each graph is in accordance with the correct sign of obtained inequality (focus on symbols > / <, without considering if the inequality is strict or large)	0.2
Either graph 1 or 2 shows: A first switch from $x = 0$ to $x = X$ that begins somewhere in the interval $t \in \left(0, \frac{\tau_1}{4}\right]$	0.2
Either graph 1 or 2 shows: The switch is instantaneous	0.2
Graph in aperiodic regime: $x = X$ for all times after the first switch	0.1
Graph in periodic regime: the behaviour is periodic with period τ_1(except for the first switch)	0.1
Graph in periodic regime: the positive and negative parts of the graph are similar	0.2
Graph in periodic regime: $x(t)/X$ is described by a rectangular function, of magnitude 1 and duty cycle 50% in steady state	0.2
Graph in periodic regime: the first step at $x = X$ last longer than others	0.1

In the real Cox's timepiece, energy provided by the mechanism is stored using a system of ratchets and used to raise a counterweight, like in a traditional clock. In the simplified model studied here, the energy recovered by the clock corresponds to the energy dissipated by the friction force exerted by the horizontal surface on the mass M. From now on, we assume that the system is dimensioned such that to work in the regime that allows the clock to recuperate energy. We also assume that the permanent regime is established. We denote W the energy dissipated by the solid friction force during a period τ_1, which can be expressed only in terms of F_s and X.

All else equal, F_s and X can be adjusted to maximize the energy W; we denote $F_s^\star$ and $X^\star$ their respective values in the optimal situation.

C.4	Considering $S_b \simeq S_c$ and $S_t \ll S_b$, determine the expressions for $F_s^\star$ and $X^\star$ as functions of ρ, g, S_c and A. Express the corresponding maximum energy $W^\star$, then calculate its numerical value with $A = 5\times 10^2\,\mathrm{Pa}$.	1pt

SOLUTION:

During a period, there is one motion to the left and one to the right. The total length of the displacement is $4X$. The total work W of the friction force is thus $W = 4F_S X$.

We have to optimize this quantity with the constraint $\xi + 2\lambda \le 2$, which can also be written as

$$\frac{2\rho g X}{A} + \frac{F_s}{S_c A} \le 1$$

The optimum is obtained at the limit of the condition, when $F_S = S_c(A - 2\rho g X)$. The work is then $W = 4X S_c(A - 2\rho g X)$. It is maximal for

$$X^\star = \frac{A}{4\rho g} \qquad \text{and} \qquad F_s^\star = \frac{A S_c}{2}$$

leading to the following optimal work

$$W^\star = \frac{A^2 S_c}{2\rho g} \simeq 20\,\mathrm{mJ}$$

MARKING SCHEME:

Starting point: $W = 4F_s X$	0.2
Optimization: $\xi + 2\lambda = 2$ or equivalent $F_s = S_c(A - 2g X)$	0.3
Expression of $X^\star$	0.1
Expression of $F_s^\star$	0.1
Expression of $W^\star$	0.2
Numerical value for $W^\star$ *with unit*: in $[19\,\mathrm{mJ}, 21\,\mathrm{mJ}]$	0.1

We denote $W_{pr}^\star$ the work of atmospheric pressure forces received by the system in the optimal situation during a period τ_1.

C.5	Express $W_{pr}^\star$, then calculate the ratio $W^\star/W_{pr}^\star$. *It could be useful to represent the evolution of the system in a (P, V) diagram, where V is the system's volume.*	1.7pt

SOLUTION:

The variations of pressure and of the vessel's position lead to fluid transfer between the cistern and the two-part tube. As a consequence, the total volume $V(t)$ occupied by the system in the atmosphere changes and can be denoted

$$V(t) = V_0 + V_1(t)$$

where V_0 is the volume in the initial state (when $x = 0$ and $P_a = P_0$) whereas $V_1(t)$ is a perturbation term. Physically, V_1 corresponds to the change of the volume of liquid in the cistern, and is thus given by

$$V_1 = \frac{m_{1,c}}{\rho}$$

where $m_{1,c}$ has already been expressed in C3 (just replace X with $x(t)$). Given that $S_b \simeq S_c$ and S_t is neglected, one obtains in any state

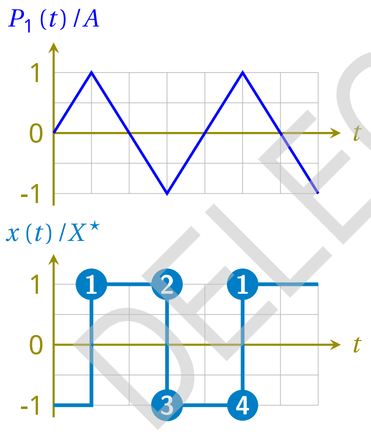
$$V_1(t) = -S_c\left[x(t) + \frac{P_1(t)}{2\rho g}\right] = -S_c X\left[\frac{x(t)}{X} + \frac{1}{\lambda}\frac{P_1(t)}{A}\right]$$

Over one period, the work of atmospheric pressure forces received by the system is defined as

$$W_{pr} = \oint_{1\text{ period}} -P_a\, dV = -\oint_{1\text{ period}} P_1\, dV_1$$

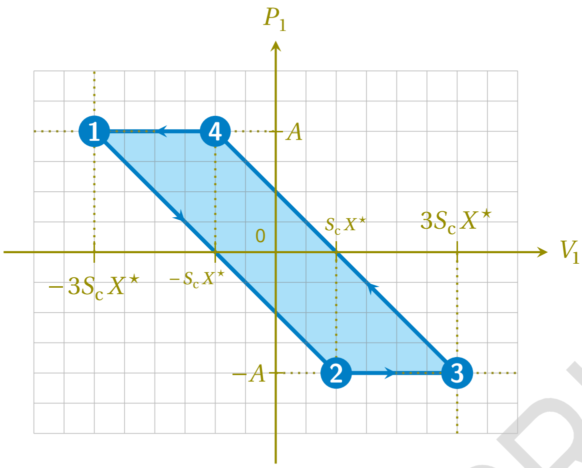
and can thus be identified to the area of the cycle described by the system in a (P_1, V_1) diagram.

Considering the optimal situation determined in the previous question, one observes the following behaviour once in steady state



State	P_1	x	V_1
1	A	$X^\star$	$-3S_c X^\star$
2	$-A$	$X^\star$	$S_c X^\star$
3	$-A$	$-X^\star$	$3S_c X^\star$
4	A	$-X^\star$	$-S_c X^\star$

Therefore, one can draw the following cycle in a (P_1, V_1) diagram



The work of the pressure force is the surface area inside this parallelogram, that is the product of its base $2\,S_c\,X^\star$ by its height $2A$. As a consequence

$$W^\star_{pr} = 4S_c\,X^\star\,A = \frac{S_c\,A^2}{\rho\,g}$$

and

$$\frac{W^\star}{W^\star_{pr}} = \frac{1}{2}$$

MARKING SCHEME:

Physical analysis: In the optimal case, the mass M switches between the two positions $x = \pm X$ when $P_1 = \pm A$	0.1
Physical analysis: During a period, the system describes a cycle formed of 2 iso-x and 2 iso-P transformations (sketch of cycle, or a table or any other pertinent description)	0.2
Physical analysis: Correct sequence of the successive states and/or direction of the cycle using x and P	0.2
General expression of the volume of the system in an (P, x) state: $V = -S_c\left[x + \frac{P_1}{2\rho g}\right] + \mathrm{Cste}$	0.3
Expressions of the volume in the 4 states of the cycle: $-3S_cX^\star \longrightarrow S_cX^\star \longrightarrow 3S_cX^\star \longrightarrow -S_cX^\star$(*all or nothing*)	0.2
Method used to calculate the work of atmospheric pressure forces: $W_{pr} = -\oint_{1\text{ period}} P_a\,dV$ (explicit integral or area of the cycle in (P, V) diagram or other pertinent method)	0.2
Obtaining $W^\star_{pr} = 4S_c\,X^\star\,A = \frac{S_c\,A^2}{\rho\,g}$	0.2
Final result: $\frac{W^\star}{W^\star_{pr}} = \frac{1}{2}$	0.3

Credits:

[1]: Bruno Vacaro;

[2]: Victoria and Albert Museum, London.

Champagne! (10 points)

Warning: Excessive alcohol consumption is harmful to health and drinking alcohol below legal age is prohibited.

Champagne is a French sparkling wine. Fermentation of sugars produces carbon dioxide (CO_2) in the bottle. The molar concentration of CO_2 in the liquid phase c_ℓ and the partial pressure P_{CO_2} in the gas phase are related by $c_\ell = k_H P_{CO_2}$, known as Henry's law and where k_H is called Henry's constant.

Fig. 1. A glass filled with champagne.

Data

- Surface tension of champagne $\sigma = 47 \times 10^{-3}\,\mathrm{J\cdot m^{-2}}$
- Density of the liquid $\rho_\ell = 1.0 \times 10^3\,\mathrm{kg\cdot m^{-3}}$
- Henry's constant at $T_0 = 20\,°\mathrm{C}$, $k_H(20\,°\mathrm{C}) = 3.3 \times 10^{-4}\,\mathrm{mol\cdot m^{-3}\cdot Pa^{-1}}$
- Henry's constant at $T_0 = 6\,°\mathrm{C}$, $k_H(6\,°\mathrm{C}) = 5.4 \times 10^{-4}\,\mathrm{mol\cdot m^{-3}\cdot Pa^{-1}}$
- Atmospheric pressure $P_0 = 1\,\mathrm{bar} = 1.0 \times 10^5\,\mathrm{Pa}$
- Gases are ideal with an adiabatic coefficient $\gamma = 1.3$

Part A. Nucleation, growth and rise of bubbles

Immediately after opening a bottle of champagne at temperature $T_0 = 20\,°\mathrm{C}$, we fill a glass. The pressure in the liquid is P_0 and its temperature stays constant at T_0. The concentration c_ℓ of dissolved CO_2 exceeds the equilibrium concentration and we study the nucleation of a CO_2 bubble. We note a its radius and P_b its inner pressure.

A.1	Express the pressure P_b in terms of P_0, a and σ.	0.2pt

SOLUTION:

A.1. Laplace's law: $P_b = P_0 + \dfrac{2\sigma}{a}$	0.2

In the liquid, the concentration of dissolved CO_2 depends on the distance to the bubble. At long distance we recover the value c_ℓ and we note c_b the concentration close to the bubble surface. According to Henry's law, $c_b = k_H P_b$. We furthermore assume in all the problem that bubbles contain only CO_2.

Since $c_\ell \neq c_b$, CO_2 molecules diffuse from areas of high to low concentration. We assume also that any molecule from the liquid phase reaching the bubble surface is transferred to the vapour.

A.2	Express the critical radius a_c above which a bubble is expected to grow in terms of P_0, σ, c_ℓ and c_0 where $c_0 = k_H P_0$. Calculate numerically a_c for $c_\ell = 4c_0$.	0.5pt

SOLUTION:

A.2.1. a_c is so $c_\ell = c_b$

A.2.2. $c_b = k_H P_b = k_H(P_0 + \frac{2\sigma}{a})$ and $c_0 = k_H P_0$ so $a_c = \dfrac{2\sigma}{P_0(c_\ell/c_0 - 1)}$

A.2.3. $a_c = 0.3\,\mu m$

A.2.1. comparison (equality $c_b = c_\ell$ or inequality $c_b \le c_\ell$)	0.1
A.2.2. exact expression $a_c = \dfrac{2\sigma}{P_0(c_\ell/c_0 - 1)}$	0.2
A.2.3. numerical value $a_c = 0.3\,\mu m$	0.2

In practice, bubbles mainly grow from pre-existing gas cavities. Consider then a bubble with initial radius $a_0 \approx 40\,\mu m$. The number of moles of CO_2 transferred at the bubble's surface per unit area and time is noted j. Two models are possible for j.

- model (1) $j = \dfrac{D}{a}(c_\ell - c_b)$ where D is the diffusion coefficient of CO_2 in the liquid.
- model (2) $j = K(c_\ell - c_b)$ where K is a constant here.

Experimentally, the bubble radius $a(t)$ is found to depend on time as shown in **Fig. 2**. Here $c_\ell \approx 4c_0$, and since bubbles are large enough to be visible, the excess pressure due to surface tension can be neglected and $P_b \approx P_0$.

A.3	Express the number of CO_2 moles in the bubble n_c in terms of a, P_0, T_0 and ideal gas constant R. Find $a(t)$ for both models. Indicate which model explains the experimental results in **Fig. 2**. Depending on your answer, calculate numerically K or D.	1.2pt

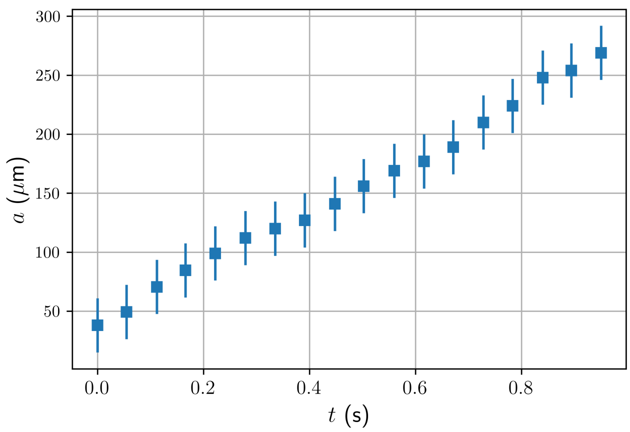


Fig. 2. Time evolution of CO_2 bubble radius in a glass of champagne (*adapted from [1]*).

SOLUTION:

A.3.1. The number of moles of CO_2 (ideal gas) inside the bubble is $n_c = \frac{4}{3}\pi a^3 \dfrac{P_0}{RT_0}$

A.3.2. Equation : balance of CO_2 in the bubble

A.3.3 $\frac{dn_c}{dt} = 4\pi a^2 \frac{da}{dt}\frac{P_0}{RT} = j4\pi a^2 \Rightarrow \frac{da}{dt} = j\frac{RT}{P_0}$

A.3.4. Model 1: $\frac{da}{dt} = \frac{DRT}{aP_0}(c_\ell - c_0)$ so $a^2 = a_0^2 + \frac{2DRT_0}{P_0}(c_\ell - c_0)t$

A.3.5. Model 2: $\frac{da}{dt} = \frac{KRT_0}{P_0}(c_\ell - c_0)$ so $a = a_0 + \frac{KRT_0}{P_0}(c_\ell - c_0)t$

A.3.6. Experimental data : $\frac{da}{dt}$ is constant: model 2

A.3.7 Slope of the experimental data : $\dot{a} \approx 150/0.62 \approx 0.24\,\text{mm}\cdot\text{s}^{-1}$

A.3.8 $K = 1.0 \times 10^{-4}\,\text{m}\cdot\text{s}^{-1}$

A.3.1. $n_c = \frac{4}{3}\pi a^3 \frac{P_0}{RT_0}$	0.1
A.3.2. any equation that that can be interpreted as a particule balance	0.1
A.3.3. equation between $\dot{a}$ (or $\dot{n}_c$) and j	0.2
A.3.4. model 1 a exact with a_0 present	0.2
A.3.5. model 2 a exact with a_0 present	0.2
A.3.6. model 2	0.1
A.3.7. value of the slope: total mark only if $\frac{da}{dt}$ is in range $[210 - 250]\,\mu\text{m}\cdot\text{s}^{-1}$	0.1
A.3.8. any value of K in range $[0.9 - 1.1] \times 10^{-4}\text{m}\cdot\text{s}^{-1}$	0.2

Eventually bubbles detach from the bottom of the glass and continue to grow while rising. **Fig. 3**. shows a train of bubbles. The bubbles of the train have the same initial radius and are emitted at a constant frequency $f_b = 20\,\text{Hz}$.

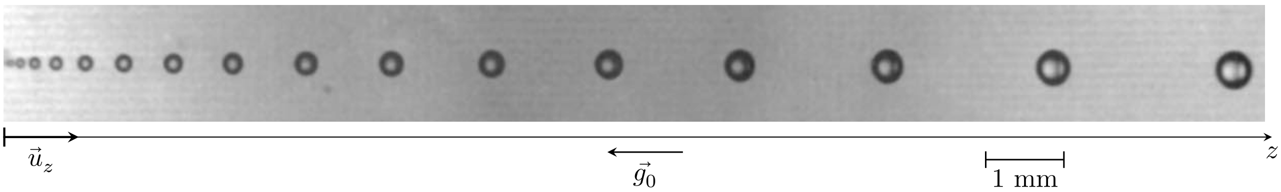


Fig. 3. A train of bubbles. The photo is rotated horizontally for the page layout *(adapted from [1])*.

For the range of velocities studied here, the drag force F on a bubble of radius a moving at velocity v in a liquid of dynamic viscosity η is given by Stokes' law $F = 6\pi\eta a v$. Measurements show that at any moment in time, the bubble can be assumed to be travelling at its terminal velocity.

A.4	Give the expression of the main forces exerted on a vertically rising bubble. Obtain the expression of $v(a)$. Give a numerical estimate of η using ρ_ℓ, g_0 and quantities measured on **Fig. 3.**	0.8pt

SOLUTION:

A.4.1. Main forces: buoyancy $\frac{4}{3}\pi a^3 \rho_\ell g_0$, drag force $6\pi\eta a v$, weight is negligible: $\frac{\rho_{CO_2}}{\rho_\ell} = \frac{P_e M_{CO_2}}{RT\rho_\ell} \approx 10^{-3}$: $m_b \ll m_\ell$

A.4.2. Simplified equation is a balance between buoyancy and drag force $\frac{4}{3}\pi a^3 \rho_\ell g_0 = 6\pi\eta a v$ so $v = \frac{2}{9\eta}a^2\rho_\ell g_0$.

A.4.3. Time between two bubbles: $\Delta t = 1/f_b$

A.4.4. Using $\eta = \frac{2\rho_\ell g_0}{9} \times \frac{a^2}{v}$ for the penultimate bubble ($n-1$) with $a_{n-1} \approx 0.19\,\mathrm{mm}$

A.4.5. $v(t_{n-1}) = \frac{z(t_n) - z(t_{n-2})}{2 \times f_\mathrm{b}^{-1}} = 4.5\,\mathrm{cm \cdot s^{-1}}$

A.4.6. $\eta \approx 2 \times 10^{-3}\,\mathrm{Pa \cdot s}$

A.4.1. Expression of main forces (gravity force present or absent): fullmark	0.1
A.4.2. expression $v = \frac{2}{9\eta} a^2 \rho_\ell g_0$ (full mark on this point with or without the gravity force)	0.2
A.4.3. taking account of the time during two positions $\Delta t = 1/f_\mathrm{b} = 5 \times 10^{-2}\,\mathrm{s}$	0.1
A.4.4. full mark for one coherent value of the radius measured on Fig.3. last bubble in $[0.20 - 0.30]\mathrm{mm}$ penultimate bubble : radius in $[0.16 - 0.24]\mathrm{mm}$ antepenultimate bubble : radius in $[0.14 - 0.22]\mathrm{mm}$	0.1
A.4.5. full mark for one coherent value of the velocity measured on Fig.3. last bubble $v \in [4.3, 4.8]\mathrm{cm \cdot s^{-1}}$ penultimate bubble $v \in [4.2, 4.6]\mathrm{cm \cdot s^{-1}}$ antepenultimate bubble $v \in [3.7 - 4.2]\mathrm{cm \cdot s^{-1}}$	0.1
A.4.6. full mark for any value or η in range $[1.0 - 4.0]10^{-3}\,\mathrm{Pa \cdot s}$	0.2

The quasi-stationary growth of bubbles with rate $q_a = \frac{\mathrm{d}a}{\mathrm{d}t}$ still applies during bubble rise.

A.5	Express the radius a_{H_ℓ} of a bubble reaching the free surface in terms of height travelled H_ℓ, growth rate $q_a = \frac{\mathrm{d}a}{\mathrm{d}t}$, and any constants you may need. Assume $a_{H_\ell} \gg a_0$ and q_a constant, and give the numerical value of a_{H_ℓ} with $H_\ell = 10\,\mathrm{cm}$ and q_a corresponding to **Fig. 2**.	0.5pt

SOLUTION:

A.5.1. $v = \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{2\rho_\ell g_0}{9\eta} a^2$ and $\frac{\mathrm{d}a}{\mathrm{d}t} = q_a$ so $\frac{\mathrm{d}z}{\mathrm{d}a} = \frac{2\rho_\ell g_0}{9 q_a \eta} a^2$

Neglecting $a(z=0)$, $z = \frac{2\rho_\ell g_0}{27 q_a \eta} a^3$ so $a_{H_\ell} = \left(\frac{27 q_a \eta H_\ell}{2\rho_\ell g_0}\right)^{1/3}$

A.5.2. $a_{H_\ell} = 3.9 \times 10^{-4}\,\mathrm{m}$ for $\eta = 2.0 \times 10^{-3}\,\mathrm{Pa \cdot s}$

A.5.1. $a_{H_\ell} = \left(\frac{27 q_a \eta H_\ell}{2\rho_\ell g_0}\right)^{1/3}$	0.3
A.5.2. full mark if $a_{H_\ell} \in [0.36 - 0.49]\mathrm{mm}$	0.2

There are N_b nucleation sites of bubbles. Assume that the bubbles are nucleated at a constant frequency

f_b at the bottom of a glass of champagne (height H_ℓ for a volume V_ℓ), with a_0 still negligible. Neglect diffusion of CO_2 at the free surface.

A.6	Write the differential equation for $c_\ell(t)$. Obtain from this equation the characteristic time τ for the decay of the concentration of dissolved CO_2 in the liquid.	1.1pt

SOLUTION:

A.6.1 The rate of bubbles reaching the free surface by unit time is $N_b f_b$

A.6.2. So the volume of CO_2 released per unit time at the free surface is:

$\frac{dV}{dt} = \frac{4}{3}\pi a_{H_\ell}^3 N_b f_b$

A.6.3. According to A.5, $\frac{dV}{dt} = \frac{18\pi N_b f_b \eta H_\ell}{\rho_\ell g_0} q_a$

A.6.4. With $q_a = \frac{da}{dt} = \frac{RT_0}{P_0} K(c_\ell - c_0)$ according to A3.

A.6.5. In the bubble, $c_b \approx c_0$. Using the ideal gas law, the total number n of CO_2 moles in V_ℓ verifies: $\frac{dn}{dt} = -\frac{P_0}{RT_0}\frac{dV}{dt} = -\frac{18\pi N_b f_b \eta K H_\ell}{\rho_\ell g_0}(c_\ell - c_0)$

With $c_\ell = \frac{n}{V_\ell}$, we get a first order linear ODE $\frac{dc_\ell}{dt} = \frac{1}{V_\ell}\frac{dn}{dt} = -\frac{18\pi N_b f_b \eta K H_\ell}{\rho_\ell g V_\ell}(c_\ell - c_0)$

A.6.6. Exponential decay with characteristic time: $\tau = \frac{\rho_\ell g V_\ell}{18\pi N_b f_b \eta K H_\ell}$

A.6.1. Correct count of bubbles reaching the free surface by unit time: $N_b f_b$	0.1
A.6.2. Balance at the free surface: $\frac{dV}{dt} = \frac{4}{3}\pi {a_{H_\ell}}^3 N_b f_b$	0.2
A.6.3. Exact expression of $\frac{dV}{dt} = \frac{18\pi N_b f_b \eta H_\ell}{\rho_\ell g_0} q_a$ using A.5.	0.1
A.6.4. $q_a = \frac{da}{dt} = \frac{RT_0}{P_0} K(c_\ell - c_0)$	0.2
A.6.5. First order linear differential equation $\frac{dc_\ell}{dt} + \frac{18\pi N_b f_b \eta K H_\ell}{\rho_\ell g_0 V_\ell}(c_\ell - c_0) = 0$. If an homogeneous mistake has been made at a previous task, but the differential equation is first order and coherent, fullmark.	0.3
A.6.6. Exponential decay with characteristic time: $\tau = \frac{\rho_\ell g_0 V_\ell}{18\pi N_b f_b \eta K H_\ell}$ full mark if the numerical coefficient is absent or different of $1/18\pi$ (reasonable solution)	0.2

Part B. Acoustic emission of a bursting bubble

Small bubbles are nearly spherical as they reach the free surface. Once the liquid film separating the bubble from the air thins out sufficiently, a circular hole of radius r forms in the film and, driven by surface tension, opens very quickly (**Fig. 4.** left). The hole opens at constant speed v_f (**Fig. 4.** right). The film outside the rim remains still, with constant thickness h.

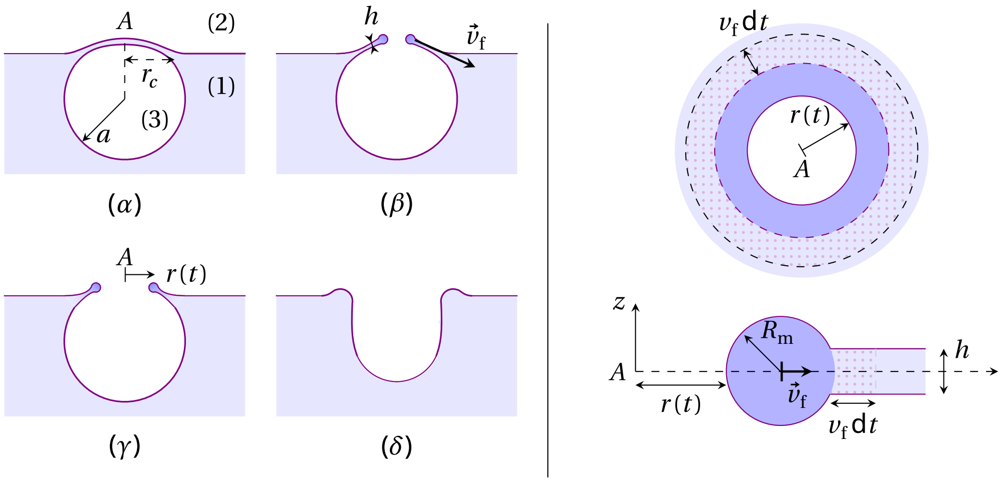


Fig. 4. *(Left)* (α) Bubble at the surface: (1) liquid, (2) air at pressure P_0 and (3), CO_2 at pressure P_b, (β) and (γ) retraction of the liquid film, where the rim is in dark blue, (δ) bubble collapse. *(Right)* Retraction of the liquid film at time t. Top: sketch of the pierced film seen from above. Bottom: cross-section of the rim and the retracting film. During $\mathrm{d}t$ the rim accumulates nearby liquid (dotted).

Due to dissipative processes, only half of the difference of the surface energy between t and $t+\mathrm{d}t$ of the rim and the accumulated liquid is transformed into kinetic energy. We further assume that the variation of the surface of the rim is negligible compared to that of the film.

B.1	Express v_f in terms of ρ_ℓ, σ and h.	1.1pt

SOLUTION:

B.1.1. and 1.2. Variation of kinetic energy: system : the rim (perimeter $\ell = 2\pi r$) and the volume $\delta V = h\ell \mathrm{d}t = h 2\pi r v_f \mathrm{d}t$ $\delta V = h\ell v_f \mathrm{d}t = h 2\pi r v_f \mathrm{d}t$: during dt the volume δV get a kinetic energy $dE_c = \frac{1}{2}\rho_\ell \delta V\, v_f^2 = \frac{1}{2}\rho_\ell h\ell v_f \mathrm{d}t = \pi r \rho_\ell h v_f \mathrm{d}t$ $dE_c = \frac{1}{2}\rho_\ell \delta V\, v_f^2 = \frac{1}{2}\rho_\ell h \ell v_f^3 \mathrm{d}t = \pi r \rho_\ell h v_f^3 \mathrm{d}t$.

B.1.3. surface tension energy: $E_s = \sigma S$ for a surface S

B.1.4. $\delta E_s = -2\sigma \ell v_f \mathrm{d}t = -4\sigma\pi r v_f \mathrm{d}t$.

B.1.5. Kinetic energy theorem: the lost energy is $\delta E_s/2 < 0$ so $\mathrm{d}E_c + \delta E_s = \delta E_s/2$

B.1.6. $v_f = \sqrt{2\sigma/\rho_\ell h}$

If partial answer: $v_f = \sqrt{\sigma/\rho_\ell h}$ obtained only by dimensional analysis: 0.2 pt to the question

B.1.1. Any expression of kinetic energy	0.1
B.1.2. Variation of kinetic energy (differential or finite variation accepted)	0.2
B.1.3. Expression of a surface energy or a variation.	0.1
B.1.4. Exact expression of δE_s	0.3
B.1.5. Kinetic energy balance (without sign mistake). If the candidate forget the energy loss, it is treated as a small mistake (-0.1pt)	0.2
B.1.6. exact expression of v_f	0.2

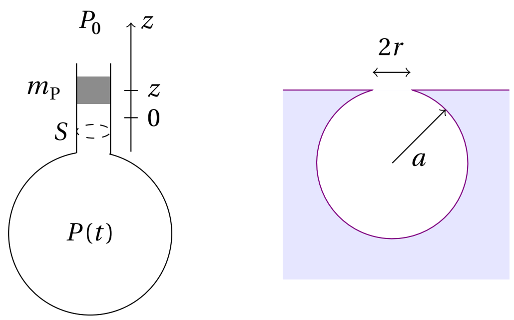


Fig. 5. *(Left)* a Helmholtz resonator. *(Right)* a bubble as an oscillator.

When the film bursts, it releases internal pressure and emits a sound. We model this acoustic emission by a Helmholtz resonator: a cavity open to the atmosphere at P_0 through a bottleneck aperture of area S (**Fig. 5**. left). In the neck, a mass m_p makes small amplitude position oscillations due to the pressure forces it experiences as the gas in the cavity expands or compresses adiabatically. The gravity force on m_p is negligible compared to pressure forces. Let V_0 be the volume of gas under the mass m_p for $P = P_0$ as $z = 0$.

B.2	Express the frequency of oscillation f_0 of m_p. Hint: for $\varepsilon \ll 1$, $(1+\varepsilon)^\alpha \approx 1 + \alpha\varepsilon$.	1.1pt

SOLUTION:

B.2.1. Pressure forces on m_p: $F_z = P(t)S - P_0 S$

B.2.2. Volume $V(t) = V_0 + Sz$

B.2.3. Adiabatic and reversible compression for an ideal gas: $PV(t)^\gamma = P_0 V_0^\gamma$ so $P(t) = P_0\left(\frac{V_0}{V_0 + Sz}\right)^\gamma = P_0\left(\frac{1}{1 + Sz/V_0}\right)^\gamma$

B.2.4. Approximation: $P(t) \approx P_0(1 - \gamma\frac{Sz}{V_0})$

B.2.5. Pressure force: $F_z = -\gamma S^2 P_0 \frac{z}{V_0}$

B.2.6. Newton's 2nd law: $m_p\ddot{z} = -\gamma S^2 P_0 \frac{z}{V_0}$ so $m_p\ddot{z} + \gamma S^2 P_0 \frac{z}{V_0} = 0$

B.2.7. Harmonic oscillator of angular frequency $\omega_0^2 = S^2 \frac{P_0\gamma}{m_p V_0}$

B.2.8. $f_0 = \frac{1}{2\pi}\sqrt{\frac{S^2 P_0 \gamma}{m_p V_0}}$

B.2.1. Pressure force with P_0	0.1
B.2.2. Expression of volume $V(t)$	0.1
B.2.3. Expression of $P(t)$ with adiabatic reversible process for an ideal gas	0.2
B.2.4. Approximate pressure	0.2
B.2.5. Exact linearized pressure force	0.1
B.2.6. Law of motion	0.1
B.2.7. Harmonic oscillator, angular frequency	0.2
B.2.8. Expression of f_0	0.1

The Helmholtz model may be used for a bubble of radius a. V_0 is the volume of the closed bubble. From litterature, the mass of the equivalent of the piston is $m_p = 8\rho_g r^3/3$ where r is the radius of the circular aperture and $\rho_g = 1.8\,\mathrm{kg\cdot m^{-3}}$ is the density of the gas (**Fig. 5**. right). During the bursting process, r goes from 0 to r_c, given by $r_c = \frac{2}{\sqrt{3}}a^2\sqrt{\frac{\rho_\ell g_0}{\sigma}}$. At the same time, the frequency of emitted sound increases until a maximum value of $40\,\mathrm{kHz}$ and the bursting time is $t_b = 3\times 10^{-2}\,\mathrm{ms}$.

B.3	Find the radius a and the thickness h of the champagne film separating the bubble from the atmosphere.	1.1pt

SOLUTION:

Determination of a

B.3.1. The maximal value of f_0 is $f_0 = 40\,\mathrm{kHz}$ is obtained for $r = r_c$

B.3.2. Exact expression of f_0 with $m = \frac{8r^3}{3}\rho_g$ and $S = \pi r_c^2$: $f_0 = \frac{1}{2\pi}\sqrt{\frac{3r_c\pi^2 P_0\gamma}{8\rho_g V_0}}$ so $f_0 = \frac{1}{2\pi}\sqrt{\frac{\gamma P_0}{\rho_g}}\sqrt{\frac{3\sqrt{3}\pi}{16a}}\sqrt{\frac{\rho_\ell g_0}{\sigma}}$

or $a = \frac{3\sqrt{3}}{64\pi}\frac{\gamma P_0}{\rho_g f_0^2}\sqrt{\frac{\rho_\ell g_0}{\sigma}}$

B.3.3. $a = 0.53\,\mathrm{mm}$

Determination of h

B.3.4. $r_c = \frac{2}{\sqrt{3}}a^2\sqrt{\frac{\rho_\ell g_0}{\sigma}}$ and $r_c = 0.15\,\mathrm{mm}$ so $v_f = \frac{r_c}{t_b} = 5.0\,\mathrm{m\cdot s^{-1}}$

B.3.5. ~~$h = \frac{2\sigma}{\rho_\ell v_f^2} = \frac{3t_b^2}{2a^4}\sqrt{\frac{\sigma^3}{\rho_\ell^3 g_0}}$~~ $h = \frac{2\sigma}{\rho_\ell v_f^2} = \frac{3t_b^2\sigma^2}{2a^4\rho_\ell^2 g_0}$

B.3.6. Numerical value $h = 3.7\,\mathrm{\mu m}$

B.3.1. Use of r_c for f_0	0.1
B.3.2. Exact expression of f_0 in terms of $a, \rho_g, \sigma, g_0, \rho_\ell, P_0$ or expression of a	0.3
B.3.3. Exact numerical value between $0.5\,\mathrm{mm}$ and $0.6\,\mathrm{mm}$	0.2
B.3.4. Relationship between t_b, v_f and r_c or a	0.2
B.3.5. Expression of h in terms of σ, ρ_ℓ and v_f (or a and t_b)	0.1
B.3.6. Numerical value $h = 3.7\,\mu\mathrm{m}$	0.2

Part C. Popping champagne

In a bottle, the total quantity of CO_2 is $n_\mathrm{T} = 0.2\,\mathrm{mol}$, either dissolved in the volume $V_\mathrm{L} = 750\,\mathrm{mL}$ of liquid champagne, or as a gas in the volume $V_\mathrm{G} = 25\,\mathrm{mL}$ under the cork (**Fig. 6.** left). V_G contains only CO_2. The equilibrium between both CO_2 phases follows Henry's Law. We suppose that the fast gaseous CO_2 expansion when the bottle is opened, is adiabatic and reversible. Ambient temperature T_0 and pressure $P_0 = 1\,\mathrm{bar}$ are constant.

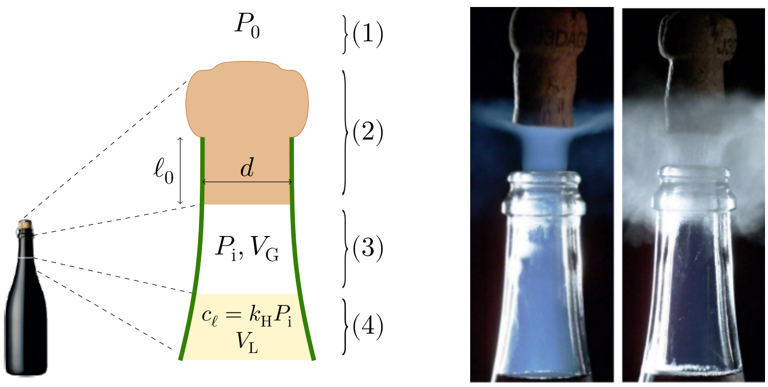


Fig. 6. *Left:* traditional bottleneck: (1) surrounding air, (2) cork stopper, (3) headspace, (4) liquid champagne. *Right*: Two phenomena observed while opening the bottle at two different temperatures (adapted from *[2]*).

C.1	Give the numerical value of the pressure P_i of gaseous CO_2 in the bottle for $T_0 = 6\,°\mathrm{C}$ and $T_0 = 20\,°\mathrm{C}$.	0.4pt

SOLUTION:

C.1.1. Conservation of CO_2 molecules: $n_T = n_V + n_L = n_V + k_H(T_0)P_i V_L$

C.1.2. Ideal gas law: $n_V = \dfrac{P_i V_G}{R T_0}$

$$P_i = \frac{n_T}{V_L k_H(T_0) + \dfrac{V_G}{R T_0}} = \frac{\dfrac{n_T R T_0}{V_G}}{1 + R T_0 k_H(T_0) \dfrac{V_L}{V_G}}$$

C.1.3. For $T_0 = 6\,°\mathrm{C}$: $P_i = 4.81\,\mathrm{bar}$

C.1.4. For $T_0 = 20\,°C$: $P_i = 7.76\,\text{bar}$

C.1.1. Conservation of CO_2 molecules	0.1
C.1.2. Litteral expression of P_i	0.1
C.1.3. For $T_0 = 6\,°C$: $P_i = 4.81\,\text{bar}$	0.1
C.1.4. For $T_0 = 20\,°C$: $P_i = 7.76\,\text{bar}$	0.1

Another step of champagne production (not described here) leads to the following values of P_i that we will use for the next questions: $P_i = 4.69\,\text{bar}$ at $T_0 = 6\,°C$ and $P_i = 7.45\,\text{bar}$ at $T_0 = 20\,°C$.

During bottle opening, two different phenomena can be observed, depending on T_0 (Fig. 6. right).

- either a blue fog appears, due to the formation of solid CO_2 crystals (but water condensation is inhibited);
- or a grey-white fog appears, due to water vapor condensation in the air surrounding the bottleneck. In this latter case, there is no formation of CO_2 solid crystals.

The saturated vapor pressure $P_{sat}^{CO_2}$ for the CO_2 solid/gas transition follows : $\log_{10}\left(\frac{P_{sat}^{CO_2}}{P_0}\right) = A - \frac{B}{T+C}$ with T in K, $A = 6.81$, $B = 1.30 \times 10^3\,\text{K}$ and $C = -3.49\,\text{K}$.

C.2	Give the numerical value T_f of the CO_2 gas at the end of the expansion, after opening a bottle, if $T_0 = 6\,°C$ and if $T_0 = 20\,°C$, if no phase transition occured. Choose which statements are true (several statements possible): 1. At $T_0 = 6\,°C$ a grey-white fog appears while opening the bottle. 2. At $T_0 = 6\,°C$ a blue fog appears while opening the bottle. 3. At $T_0 = 20\,°C$ a grey-white fog appears while opening the bottle. 4. At $T_0 = 20\,°C$ a blue fog appears while opening the bottle.	0.7pt

SOLUTION:

C.2.1. The adiabatic reversible expansion goes from P_i to P_0.

C.2.2. $T_f = T_0\left(\frac{P_i}{P_0}\right)^{(1/\gamma)-1}$

C.2.3. For $T_0 = 6\,°C$: $P_i = 4.69\,\text{bar}$ and $T_f = 195.3\,\text{K} = -77.8\,°C$.

C.2.4. For $T_0 = 20\,°C$: $P_i = 7.45\,\text{bar}$ and $T_f = 184.3\,\text{K} = -88.8\,°C$.

C.2.5. *First method:* comparison $P_{sat}(T_f)$ and $P_f = P_0$.

Second method: evaluation of the transition temperature at P_0 and comparison with T_f.

C.2.6. *First method:* $P_{sat}^{CO_2}(T_f = 6\,°C) = 1.07\,\text{bar} > P_0$. As the solid-liquid frontier has a positive slope in P, T state-diagram, the final state of CO_2 is gaseous. $P_{sat}^{CO_2}(T_f = 20\,°C) = 0.41\,\text{bar} < P_0$. As the solid-gas frontier has a positive slope in P, T state-diagram, the final gaseous state hypothesis is inconsistent and a phase transition has occured in the latter case.

Second method: $T_{trans} = \frac{B}{A - \log_{10}\left(\frac{P_0}{P_0}\right)} - C$. $T_{trans} = 194.4\,\text{K} = -78.8\,°C$. For $T_0 = 6\,°C$: $T_f = 195.3\,\text{K} > T_{trans}$; the final state of CO_2 is gaseous. For $T_0 = 20\,°C$: $T_f = 184.3\,\text{K} < T_{trans}$; the final gaseous state hypothesis is inconsistent and a phase transition has occured.

C.2.7. The true statements are: 1 and 4.

C.2.1. Final pressure of the expansion.	0.1
C.2.2. Litteral expression of T_f.	0.1
C.2.3. For $T_0 = 6\,°\text{C}$: $P_i = 4.69\,\text{bar}$ and $T_f = 195.3\,\text{K}$;	0.1
C.2.4. For $T_0 = 20\,°\text{C}$: $P_i = 7.45\,\text{bar}$ and $T_f = 184.3\,\text{K}$;	0.1
C.2.5. Idea of comparison between P_{sat} and P_0 or evaluation of the transition temperature at P_0 and idea of comparison with T_f.	0.1
C.2.6. Numerical comparison.	0.1
C.2.7. True statements (all or nothing).	0.1

During bottle opening, the cork stopper pops out. We now determine the maximum height H_c it reaches. Assume that the friction force F due to the bottleneck on the cork stopper is $F = \alpha A$ where A is the area of contact and α is a constant to determine. Initially, the pressure force slightly overcomes the friction force. The cork's mass is $m = 10\,\text{g}$, its diameter $d = 1.8\,\text{cm}$ and the length of the cylindrical part initially stuck in the bottleneck is $\ell_0 = 2.5\,\text{cm}$. Once the cork has left the bottleneck, you can neglect the net pressure force.

C.3	Give the numerical value of H_c if the external temperature is $T_0 = 6\,°\text{C}$.	1.3pt

SOLUTION:

C.3.1. Let us evaluate the work of the friction force. $\vec{F} = -\alpha.\pi d(l_0 - z)\vec{u_z}$. Initially, this force slightly compensates the pressure force: $F = \pi\alpha d\ell_0 = \pi\dfrac{d^2}{4}(P_i - P_0)$ so $\alpha = (P_i - P_0)\dfrac{d}{4\ell_0}$

C.3.2. $\vec{F} = -(P_i - P_0)\pi d^2 \dfrac{(\ell_0 - z)}{4\ell_0}\vec{u_z}$ The total work is therefore: $W_f = -\alpha\pi d\dfrac{\ell_0^2}{2} = -\dfrac{(P_i - P_0)\pi d^2}{8}\ell_0$

C.3.3. and C.3.4. Work of the internal pressure force:

First method: the variation of internal energy of the gas is:

$$\Delta U_g = \frac{n_V R}{\gamma-1}(T_f - T_0) = \frac{n_V R}{\gamma-1}T_0\left(\frac{1}{\left(1+\frac{\pi d^2\ell_0}{4V_G}\right)^{\gamma-1}} - 1\right) = \frac{P_i V_G}{\gamma-1}\left(\frac{1}{\left(1+\frac{\pi d^2\ell_0}{4V_G}\right)^{(\gamma-1)}} - 1\right)$$

As its expansion is adiabatic: $\Delta U_g = W_{\text{cork}\to\text{CO2}} = -W_{\text{CO2}\to\text{cork}}$ The cork stopper receives therefore a work from this gas equals to $-\Delta U_g$.

$$W_{\text{CO2}\to\text{cork}} = \frac{P_i V_G}{\gamma-1}\left(1 - \frac{1}{\left(1+\frac{\pi d^2\ell_0}{4V_G}\right)^{(\gamma-1)}}\right)$$

Second method: let us write P the internal pressure during the expansion. The work received by the cork is:

$W_{\text{CO2}\to\text{cork}} = \int_{V_G}^{V_F} P\,dV$, where $V_F = V_G + \dfrac{\pi d^2\ell_0}{4}$ and $P_0 V_F^\gamma = P_i V_G^\gamma$.

The integration leads to the same result.

C.3.5. The work due to the external pressure P_0 is: $W_e = -P_0.\frac{\pi d^2}{4}\ell_0$

C.3.6. Energy balance. The cork stopper has an initial kinetic energy: $E_c = -\Delta U_g + W_f + W_e$

(The work of the weight is negligible and should not be taken into account).

At $T_0 = 6\,°\mathrm{C}$: $P_i = 4.69\,\mathrm{bar}$. $W_f = -1.17\,\mathrm{J}$; $W_e = -0.64\,\mathrm{J}$; $\Delta U_g = -2.57\,\mathrm{J}$; $E_c = 0.76\,\mathrm{J}$

C.3.7. The maximum height reached by the cork stopper is therefore: $H_c = \frac{E_c}{mg_0} = \frac{-\Delta U_g + W_f + W_e}{mg_0}$.

C.3.8. $H_c = 7.7\,\mathrm{m}$

If the candidates assumed a constant pressure P_i for the gaseous CO_2 during its expansion, they would find a work done by CO_2on the cork equal to: $P_i(\pi\ell_0 d^2/4) = 3$ J instead of $2.56\,\mathrm{J}$ and finally $H_c = 12\,\mathrm{m}$. The difference is not negligible!

C.3.1. Correct expression of α (all or nothing). If α is not correct (contribution of P_0 forgotten for example), 0 point but the following items are evaluated with this uncorrect α.	0.2
C.3.2. Expression of the friction work (all or nothing)	0.2
C.3.3. Consequences of the adiabatic reversible expansion (1st principle with $Q = 0$ or $PV^\gamma = P_i V_G^\gamma$)	0.1
C.3.4. Exact expression of the work (all or nothing) *Partial points :* if P is considered constant during the expansion, 0 point for C.3.4. but all points for the following items if coherent with the incorrect work expression.	0.3
C.3.5. Work due to external pressure correct. If this item is forgotten by the candidate, 0 point.	0.1
C.3.6. Correct E_c with the 3 contributions (even if errors in the writing of the contributions). If the candidate has forgotten the contribution of the external pressure, 0 point.	0.1
C.3.7. Correct energy balance during the free flight or use of Newton's second law.	~~0.2~~0.1
C.3.8. Correct numerical value of H_c. If the candidate has forgotten the contribution of the external pressure in C.3.5 but H_c is coherent, fullmark.	0.2

[1] Liger-Belair *et al,* Am. J. Enol. Vitic., Vol. 50, No. 3 (1999).

[2] Liger-Belair *et al.*, *Sc. Reports* **7**, 10938 (2017).