Earth's magnetic field measurement (10 points)

Introduction

This problem aims to measure the horizontal component of the Earth's magnetic field. A magnet will first be characterized using a so called Gouy balance, before being used to measure this magnetic field.

In the entire problem, uncertainties are expected to be determined only from the fits and not from the individual experimental points.

Equipment list

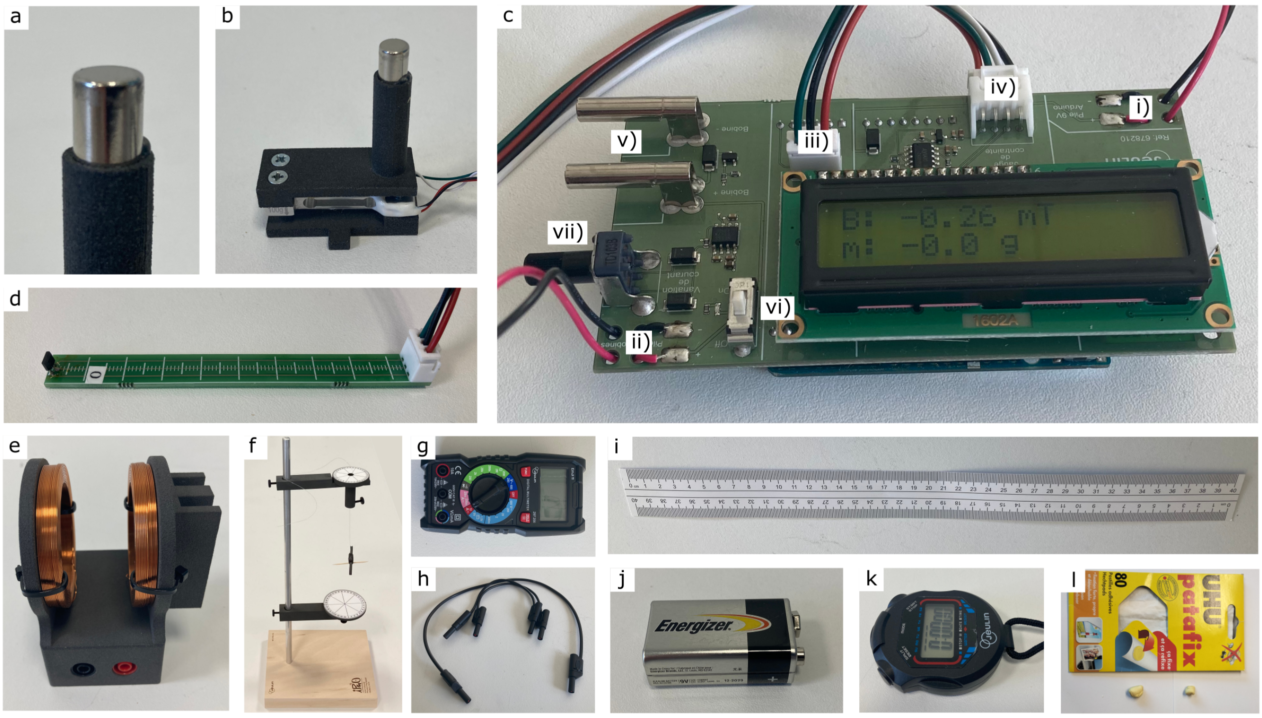


Fig. 1. Photographs of all equipment.

The list of equipment is given below and illustrated in Fig. 1. The number of items is indicated between [] when it is greater than one. Students should ask for help if something appears not to be working.

- **(a)** Magnets [3]. One magnet is attached to the force sensor (b) and should not be removed. Another magnet is inserted into the pod (f) and should not be removed until specified. The last one will be used in A.5. All magnets are supposed identical.
- **(b)** Force sensor. Connected to the Arduino (c), this sensor measures the force along its axis, noted m_f, in grams-force ("g"), which is the force experienced by a 1-gram mass on the earth's surface in the gravity field ($g_0 = 9.81\,\mathrm{m \cdot s^{-2}}$) . One of the magnets (a) is attached to it. Each time it is switched back on, the sensor display is reset to 0, regardless of the situation. *This sensor must not be subjected to forces in excess of 200 grams. It needs to be unpacked carefully.*
- **(c)** Arduino with digital display. This element is used to power the coils (e) and to perform force and magnetic field measurements, displayed directly in gram-force ("g") and mT. The battery (j) powering the Arduino must be connected to slot (i), and the battery (j) powering the coils (e) to slot (ii) (pay attention to connection polarity). The force sensor (b) and magnetic field sensor (d) should be connected to slots (iv) and (iii) respectively, and the coil power cables to slots (v). A switch (vi) closes the coil supply circuit (indicated by an LED), whose electric current can be controlled in (vii).

- **(d)** Magnetic field sensor with ruler. Connected to the Arduino (c), this probe measures the field B_z along the direction $\vec{e_z}$ of the ruler, in mT.
- **(e)** Coils in anti-Helmholtz configuration (wound in opposite directions). These coils must be connected in series with the ammeter (g) and to the Arduino (c) to create a magnetic field.
- **(f)** Metallic stand on a wooden base, with suspended pod where a magnet (a) is initially inserted, and with angle markers. The detailed assembly of this device is explained below.
- **(g)** Multimeter. Only used as an ammeter at the $10\,\mathrm{A}$ range. If left inactive, the multimeter switches off, and must be switched back on by returning it to the “OFF” position. Do not use the two cables supplied in the multimeter case.
- **(h)** Electric wires [3].
- **(i)** $40\,\mathrm{cm}$ ruler.
- **(j)** $9\,\mathrm{V}$ batteries [3]. Their capacity is of the order of $300\,\mathrm{mA \cdot h}$.
- **(k)** Chronometer.
- **(l)** Adhesive paste. Can be used for the entire problem.

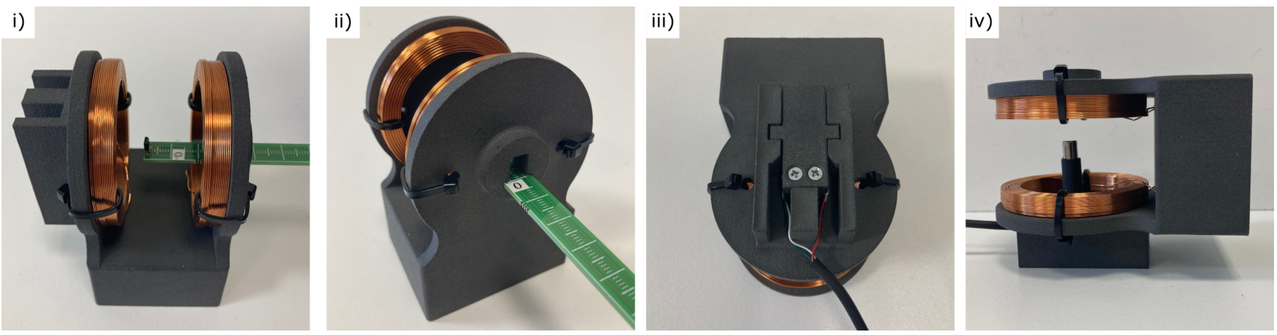


Fig. 2. Use of sensors inside the anti-Helmholtz coils.

Use of sensors interfaced with the Arduino (Fig. 2)

The magnetic field sensor (d) can slide in the coils (e) as shown in (i), while measuring the field on their axis. The $z = 0$ position for the sensor is shown in (ii), and z increases as it moves inside the coils.

The force sensor (b) is inserted into the coils as shown in (iii), before turning the coil as in (iv) so that the transducer is vertical. *To do this, be sure to route the electrical wires through the gutters provided.*

Installation of equipment (f) (Fig. 3), to be mounted only before starting part B, with a $34\,\mathrm{cm}$ wire

- Insert the metal post (f0a) into the wooden plate with plastic feet (f0b) to form the stand (f0).
- The part (f1) is located on the lower part and marks the angle of the pod. Install the arm (f1b) on the metal post by means of a screw (f4), then fix the part (f1a) on it with a second screw (f4).
- The part (f2) is located on the upper part and hold the wire supporting the pod. Install the arm (f2b) on the metal post by means of a screw (f4), then insert the part (f2a) on it.
- To build the pod (f3), insert the inertia bar (f3b) and a toothpick (f3c) into the carrier part (f3a) on which a magnet (a) is already inserted. Insert the wire supporting the pod into the part (f2a), and secure it with a screw (f4). Turning part (f2a) changes the angle at which the wire is attached. The toothpick allows to precisely measure the angular position of the pod.

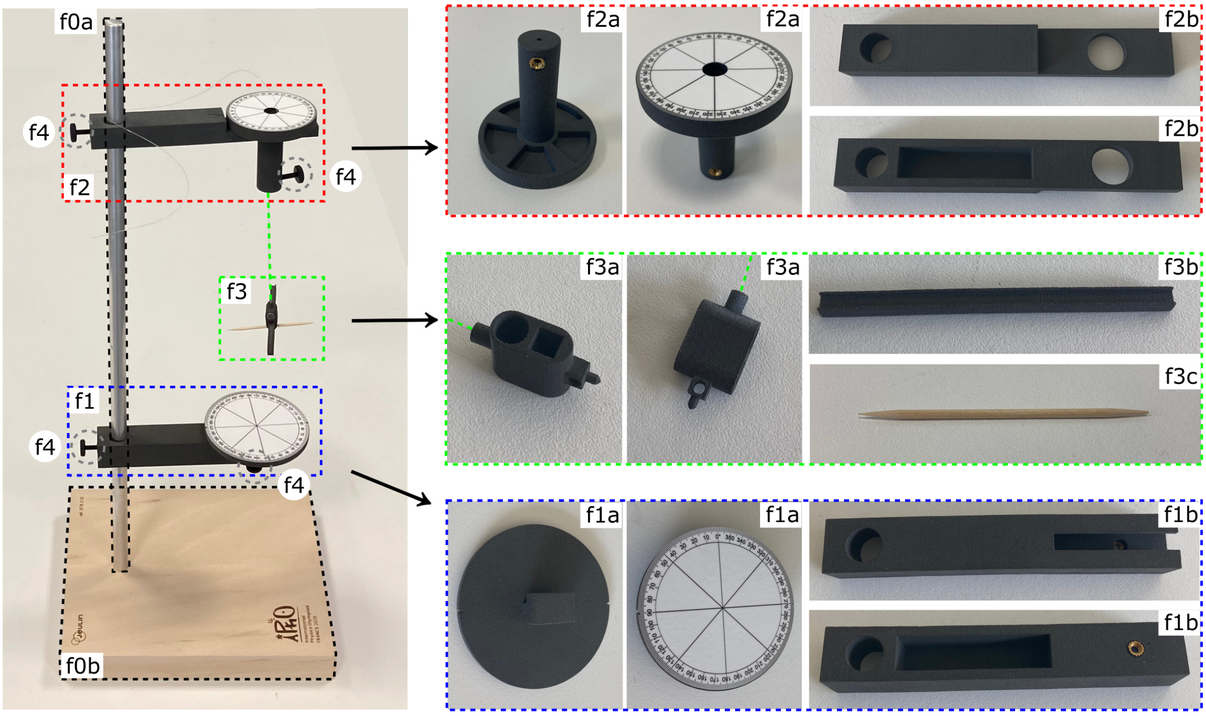


Fig. 3. Installation of the pod on the metallic stand. Parts (f1a), (f1b), (f2a), (f2b), and (f3a) are shown from two different angles. There are four identical (f4) plastic screws.

SOLUTION:

Please note that the numerical results given in the solution come from a single, consistent measurement session. The measurement ranges used in the notation take into account several measurements by various testers.

Marking scheme: students are not penalized for forcing a linear fit to pass through the origin when the studied law is proportional.

Part A. Gouy balance and magnetic moment

Modeling

We assume that a magnet can be treated as a magnetic dipole of magnetic moment $\vec{m}_{\mathrm{m}}$. The force experienced by such a dipole of magnetic moment $\vec{m}_{\mathrm{m}} = m_{\mathrm{m}}\vec{e}_z$ in a magnetic field $\vec{B} = B(z)\vec{e}_z$ is

$$\vec{F}(z) = m_{\mathrm{m}}\frac{dB(z)}{dz}\vec{e}_z\,. \tag{1}$$

When an electric current i flows through the anti-Helmholtz coils, the field $\vec{B}$ along the unit vector $\vec{e}_z$ of revolution axis is

$$\vec{B}(z) = \alpha\, i(z - z_0)\vec{e}_z\,. \tag{2}$$

This equation is only valid near the center of the device, denoted by $z = z_0$.

Magnetic field in the coils

A.1	Estimate numerically the typical operating time τ of one of the batteries used in the experiment, with an electric current of the order of $2\,\mathrm{A}$.	0.2pt

SOLUTION:

The 9V battery capacity is $Q = I \cdot \Delta t = 300$ mA·h. Using an electric current $I = 2$A, the time of use is $0.3 \times 3600/2 \approx 540\mathrm{s} \approx 9$min.

A.1.1. One value in the intervalle $6 \leq \tau \leq 12$ min or ($360 \leq \tau \leq 720$ s).	0.2

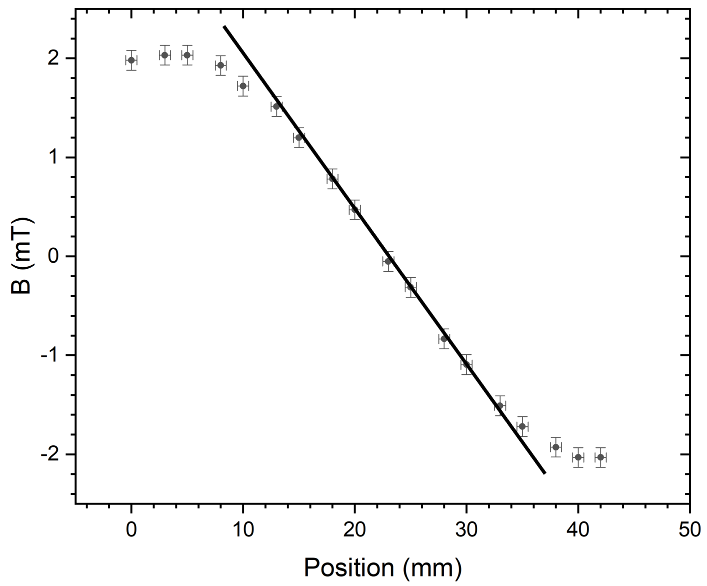
This result must be taken into account when developing the protocols later on, knowing that the coils are only used in part A. Note that a spare battery is available if required.

Insert the magnetic field sensor into the coils, as shown in Fig 2. See also this figure for the identification of the sensor position in the coils.

A.2	At a fixed electric current $i_0 \simeq 1.0\mathrm{A}$, measure and plot the magnetic field B_z as a function of the position z of the sensor on the axis of the coils. Identify the largest region $[z_{\mathrm{min}}, z_{\mathrm{max}}]$ where the magnetic field is experimentally linear with respect to position.	0.8pt

SOLUTION:

The plot below is obtained at $i_0 = 1.0$A. At the centre of the device is a zone in which the field is a linear function of position. At the edges of the device, you can see the saturation of the classical field as you approach the two coils. The zone of linearity from the figure is [0.015 ; 0.032] m.

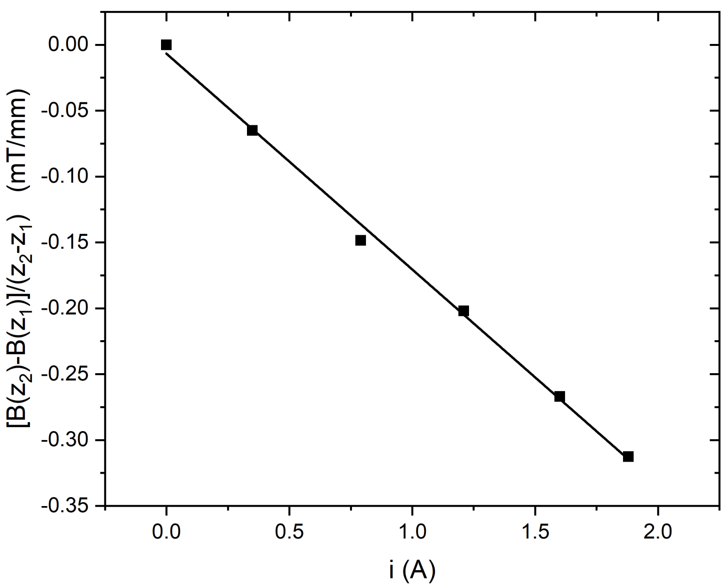


A.2.1: Measure 6 points or more. B in [-12 ; 12] mT and z in [0 ; 60] mm.	0.1
A.2.2: Measure 8 points or more. B in [-12 ; 12] mT and z in [0 ; 60] mm.	0.1
A.2.3: Plot (axes, units).	0.1
A.2.4: Experimental plot showing linearity with correct sampling. At least 5 points.	0.1
A.2.5: Experimental plot showing deviation from linearity on the left side in [12 ; 18]mm.	0.1
A.2.6: Experimental plot showing deviation from linearity on the right side in [29 ; 35]mm.	0.1
A.2.7: z_{min} in [12 ; 18] mm.	0.1
A.2.8. z_{max} in [29 ; 35] mm.	0.1

A.3	By placing the sensor at two positions (z_1, z_2) in this region of linear dependency, draw a curve to verify the electric current dependency of $\vec{B}$ given by equation (2), and determine the value of α, with its uncertainty.	0.9pt

SOLUTION:

The centre of the linear zone is around 23 mm. The values of the magnetic field at two positions $z_1 = 13\text{mm}$ and $z_2 = 33\text{mm}$ are measured for several electric current values. This allows to compute the gradient $\frac{B(z_2)-B(z_1)}{z_2-z_1}$ of the magnetic field as a function of electric current.



We have a linear evolution. The possible residual y-intercept may be due to the fact that the sensor is not correctly calibrated. A typical value for the slope gives $\alpha = 0.150 \pm 0.007\,\text{T}\cdot\text{m}^{-1}\cdot\text{A}^{-1}$.

A.3.1: 3 measures or more of B at 2 positions (total 6). B in [-25 ; 25] mT and I in [-3 ; 3] A.	0.1
A.3.2: 5 measures or more of B at 2 positions (total 10). B in [-25 ; 25] mT and I in [-3 ; 3] A.	0.1
A.3.3: Plot (axes, units).	0.1
A.3.4: Identification and calculation of the relevant slope quantity. Either $B(z)/(z-z_0)$ or $(B(z_2)-B(z_1))/(z_2-z_1)$, or these quantities divided by i.	0.1
A.3.5: Experimental plot showing linearity with correct sampling.	0.1
A.3.6: α value (with units) in [0.11 ; 0.19] T/m/A,	0.1
A.3.7. α value (with units) in [0.13 ; 0.17] T/m/A.	0,1
A.3.8. $\delta\alpha$ value (with units) in [0.001 ; 0.02] T/m/A.	0.1
A.3.9. $\delta\alpha$ value (with units) in [0.003 ; 0.01] T/m/A,	0,1

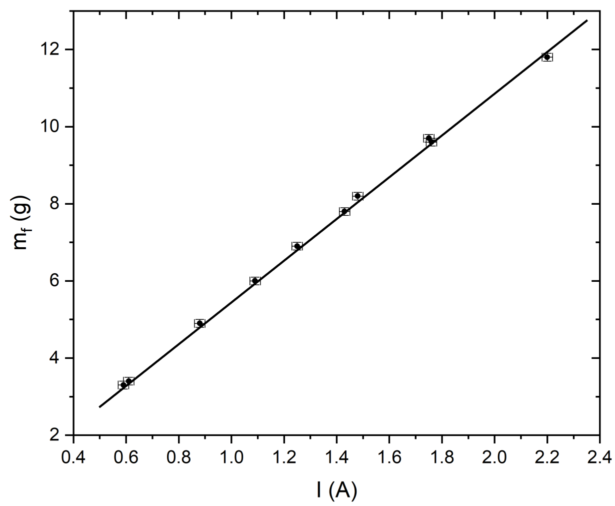
Gouy balance

Remove the magnetic field sensor from the coils, and carefully place the force sensor inside, as described in Fig. 2, with particular attention to the placement of electrical wires in the gutters.

A.4	Perform experimental measurements of the gram-force m_f as a function of current i. Draw an appropriate plot to determine the value of the magnetic moment m_m of the magnet, with its uncertainty.	0.8pt

SOLUTION:

We vary the electric current i and measure the effective mass, which gives



The slope of the curve is 5.48 gram-force/A, giving a slope of $(53.7 \pm 0.8) \times 10^{-3}$ N/A. Finally, the magnetic moment is $m_m = \dfrac{53.7 \times 10^{-3}}{0.150} = 0.358\,\mathrm{A \cdot m^2}$. The uncertainty is obtained from $\dfrac{\delta m}{m} = \dfrac{\delta p}{p} + \dfrac{\delta \alpha}{\alpha} = 0.05$.

The magnetic moment $m_m = 0.36 \pm 0.02\,\mathrm{A \cdot m^2}$.

A.4.1: 6 measures or more. m_f in [-20 ; 20] g and I in [-3 ; 3] A.	0.1
A.4.2: 8 measures or more. m_f in [-20 ; 20] g and I in [-3 ; 3] A.	0.1
A.4.3: Plot (axes, units).	0.1
A.4.4: Experimental plot showing linearity with correct sampling.	0.1
A.4.5: m_m value (with units) in [0.25 ; 0.45] A.m^2.	0.1
A.4.6: m_m value (with units) in [0.30 ; 0.40] A.m^2	0.1
A.4.7. δm_m value (with units) in [0.003, 0.07] A.m^2.	0.1
A.4.8 δm value (with units) in [0.01, 0.03] A.m^2.	0.1

Measurements of force in newton (N) are accepted.

Alternative measurement of the magnetic moment

In the dipolar approximation, the magnetic field of a magnet of magnetic moment m_m on its revolution axis z is

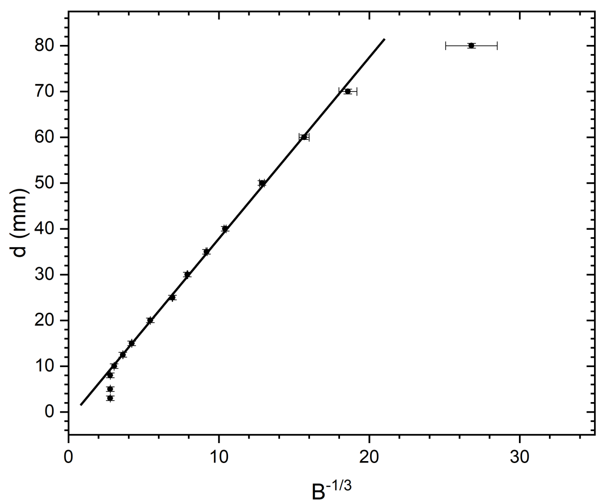
$$B_z(z) = \frac{\mu_0 m_\mathrm{m}}{2\pi(z - z_\mathrm{a})^3}, \tag{3}$$

where z_a is not necessarily the geometric center of the magnet, and where $\mu_0 = 4\pi 10^{-7}\,\mathrm{H\cdot m^{-1}}$.

A.5	Measure the magnetic field B_z along the revolution axis of the free magnet, as a function of distance z. Draw a curve to verify the model given Eq. (3), showing its experimental deviations. Deduce a new value for m_m, with uncertainty.	1.3pt

SOLUTION:

The field B is measured directly by sticking the third magnet on the graduated ruler. You can also use the Hall sensor directly. The measurements are shown below, where the position is plotted as a function of $B^{-1/3}$ (see figure below).



The slope is $\left(\frac{\mu_0 m_m}{2\pi}\right)^{1/3} = (4,1 \pm 0.1) \times 10^{-3}$ m $\cdot$ T$^{1/3}$, and then the new value of the magnetic moment is $m_m = 0,31 \pm 0.01$ A $\cdot$ m^2.

A.5.1: 6 measures or more. B in [-100 ; 100] mT and d in [0 ; 40] cm.	0.1
A.5.2: 8 measures or more. B in [-100 ; 100] mT and d in [0 ; 40] cm.	0.1
A.5.3: Plot (axes, units).	0.1
A.5.4: Identification and calculation of the relevant quantity. Either $z = f(B^{-1/3})$ or related quantity.	0.2
A.5.5: Identification of the valid region, out of near field (small z).	0.1
A.5.6. Identification of the valid region : not limited by digital quantification (high z).	0.1
A.5.7: Experimental plot showing linearity with correct sampling.	0.2
A.5.8: m_m value (with units) in [0.25 ; 0.45] A.m^2.	0.1
A.5.9: m_m value (with units) in [0.30 ; 0.40] A.m^2.	0.1
A.5.10: δm_m value (with units) in [0.001, 0.05] A.m^2	0.1
A.5.11: δm_m value (with units) in [0.005, 0.02] A.m^2.	0.1

A.6	Given the two results obtained in A.4 and A.5, propose a final experimental value of m_m with its uncertainty.	0.2pt

SOLUTION:

The final value is given by the averaged value of the previous measurements, so $m_m = 0,33 \pm 0,01$ A $\cdot$ m^2.

A.6.1 m_m value (with units) in [0.30 ; 0.40] A.m^2.	0.1
A.6.2 δm_m value (with units) in [0.005, 0.03] A.m^2.	0.1

Part B. Determining the earth's magnetic field

Modeling

We now study the oscillating motion of the magnet in a horizontal plane to estimate the value of the horizontal component B_e of the Earth's magnetic field, see Fig. 3 and the assembly instructions above Fig.3. The pod (f3), containing the magnet, is subjected to two torques around the vertical axis:

- the torque of the wire, modeled as $\Gamma_f = -\frac{C_f}{L}(\theta - \theta_0)$, where C_f is a constant and L the total length between the two attachments of the wire, and θ_0 corresponds to the angle for which the wire is not twisted,
- the torque of the Earth's magnetic fields, given by $\Gamma_e = -m_m B_e \sin(\theta - \theta_e)$, when the angular position of the Earth's magnetic field is given by the angle θ_e.

Denoting J the unknown moment of inertia of the pod and magnet assembly around the vertical axis, the angular momentum theorem gives

$$J\frac{d^2\theta}{dt^2} = \Gamma_f + \Gamma_e = -\frac{C_f}{L}(\theta - \theta_0) - m_m B_e \sin(\theta - \theta_e). \quad (4)$$

When the $\sin(\theta - \theta_e) \simeq \theta - \theta_e$ approximation is valid, this leads to an sinusoidal oscillation at a period T. For this part, adhesive past (l) is moldable into any shape or size and attachable to other devices.

Caution: To avoid disturbance from external magnetic fields, the magnet must be placed at least $20\,\mathrm{cm}$ away from any metal object or magnetic source (including the other magnets).

Experimental set-up and first measurement

For questions B.1 to B.5, set the length of the wire to $L = 34\,\mathrm{cm}$ and make sure that it is not twisted. In this setting, we begin by assuming that the torque from the wire is negligible with respect to the torque from the Earth's magnetic field, a hypothesis to which we will return later.

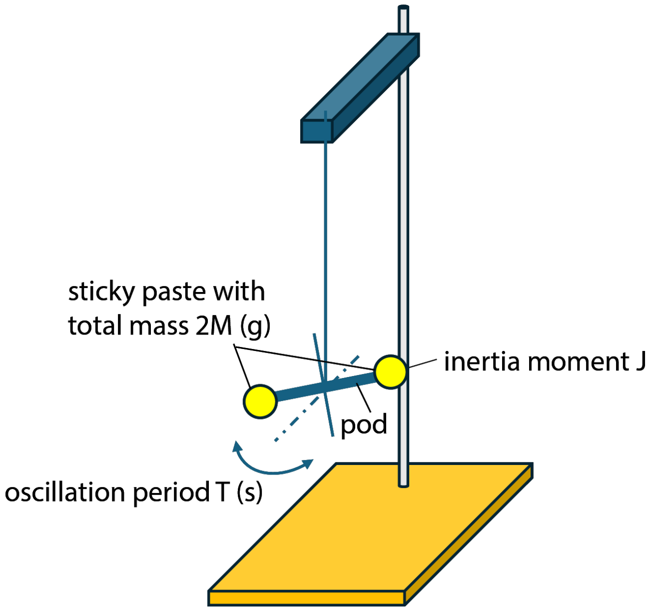
To align θ_0 with θ_e, use piece (f2a) to adjust θ_0 so the pod (f3) does not rotate when the magnet is removed. Then reinsert the magnet in the pod, and keep θ_0 unchanged until question B.5.

B.1	Propose an experimental protocol to determine B_e. Introduce the different quantities you will measure and their units. Depict these quantities on a detailed schematic, and relate them to those given in the instructions through an equation. For each quantity, specify whether it is fixed (F) or varies (V) throughout the protocol.	0.3pt

SOLUTION:

The figure below describes the proposed experiment. The period T for small oscillations is measured (with best precision using several periods). Since the inertial moment J_0 of the pod is unknown, adding sticky paste to both ends of the pod allows the change of the inertial moment $J = J_0 + \Delta J$. The length of the pod arm is $r_a = 0.04\,\mathrm{m}$.

The differential equation verified by the pod at small angles is $\frac{\Delta J}{m_m} = \frac{1}{\omega^2} B_e - \frac{J_0}{m_m}$. The period is $T = \frac{2\pi}{\omega}$ and the variation of inertial moment by adding a total mass of sticky paste $2m_a$ is $\Delta J = 2m_a . r_a^2$. Therefore $\frac{2m_a r_a^2}{m_m} = \frac{T^2}{4\pi^2} B_e - \frac{J_0}{m_m}$.

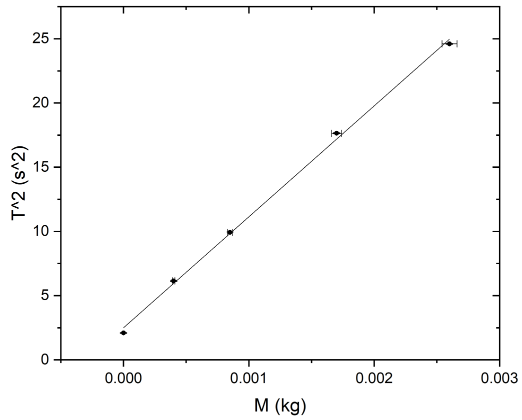


B.1.1: Period T, in second, and expression of T as a function of other quantities.	0.1
B.1.2 Added mass m_a, in gram, with schematic.	0.1
B.1.3 Radius of the added mass r_a, in centimeter, with schematic.	0.1

Note that the grading scheme will be evaluated as follows to take into account alternative protocols: a) If the students propose any of the quantities that already appear in the grading scheme, they will have the related points. b) If the students propose any protocol that works, and properly introduce the relevant quantities, they will have the entire points for the questions. c) No point will be given for additional quantities that do not correspond to a working protocol.

B.2	Using the protocol described above, draw a graph to determine a first value of B_e, with its uncertainty.	1.1pt

SOLUTION:



Since $\frac{2m_a r_a^2}{m_m} = \frac{T^2}{4\pi^2}B_e - \frac{J_0}{m_m}$, plotting T^2 versus m_a should give a linear function of slope $p = \frac{8\pi^2 r_a^2}{m_m B_e}$.
One finds a slope $p = 8640 \pm 200$ A$^{-1}\cdot$T^{-1}. Using the value $m_m = 0,31 \pm 0.01$ A$\cdot$m^2, one obtains $B_e = 47\ \mu$T. the relative uncertainty is $\frac{200}{8640} + \frac{0,01}{0,305} \approx 0,06$. Therefore $B_e = (47 \pm 3)\ \mu$T.

B.2.1: 4 measures or more.	0.1
B.2.3: 6 measures or more.	0.1
B.2.3 : Identification and calculation of the relevant quantity. Either $T^2 = f(J_a)$ or $f(m_a r_a^2)$ or related quantity.	0.2
B.2.4 : Plot (axes, units).	0.1
B.2.5: Experimental plot showing linearity with correct sampling.	0.2
B.2.6 B_e value (with units) in [1 ; 10] 10^{-5} T.	0.1
B.2.7. B_e value (with units) in [1.5 ; 7] 10^{-5} T.	0.1
B.2.8. δB_e value (with units) in [0.1 ; 1] 10^{-5} T.	0.1
B.2.9. δB_e value (with units) in [0.2 ; 0.5] 10^{-5} T.	0.1

Evaluation of the torque from the wire

B.3	Keeping $L = 34\,\text{cm}$, study the motion of the pod without the magnet, and determine the value of C_f, with experimental uncertainty: perform one period measurement for two system configurations. Specify the equation relating C_f to the measured quantities.	0.7pt

SOLUTION:

The magnet is removed. The period of oscillation therefore depends on the torque due to the twisting of the wire and the moment of inertia. As the moment of inertia of the cradle remains unknown, we can measure the period T_1 without sticky paste and the period T_2 with sticky paste, for a fixed length of wire. The equations involved are

$$J = \frac{C_f T_1^2}{4\pi^2 L},$$

$$J + 2m_a r_a^2 = \frac{C_f T_2^2}{4\pi^2 L}.$$

Taking two measures, for $m_a = 0$ and $m_a = 2{,}6$ g, one finds $T_1 = 4.2 \pm 0.2$ s and $T_2 = 15.8 \pm 0.2$ s, and therefore $C_f = \frac{8\pi^2 L m_a r_a^2}{T_2^2 - T_1^2}$, so $C_f = (5.1 \pm 0.2) \times 10^{-7}$ N · m²/rad (m³ · kg · s⁻²).

B.3.1: Choice of a parameter that varies. J_a through m_a and/or r_a.	0.1
B.3.2: Measurements of T_1 and T_2.	0.1
B.3.3: Expression of C_f as a function of measured quantities.	0.1
B.3.4: C_f value (with units) in [2 ; 10] 10^{-7}N.m²/rad.	0.1
B.3.5: C_f value (with units) in [3 ; 8] 10^{-7}N.m²/rad.	0.1
B.3.6: δC_f value (with units) in [0.05 ; 1.0] 10^{-7}N.m².rad^{-1}	0.1
B.3.7: δC_f value (with units) in [0.1 ; 0.5] 10^{-7}N.m².rad^{-1}	0.1

B.4	Using previous measurements, give the expression and determine numerically the critical length L_c for which the amplitude factors C_f/L and $m_m B_e$ of the Γ_f and Γ_e torques are equal. In question B.2, what was the ratio $(C_f/L)/(m_m B_e)$? Choose from the intervals: [0 %, 1 %[; [1 %, 5 %[; [5 %, 20 %[; [20 %, 50 %[; [50 %, ∞%[.	0.3pt

SOLUTION:

The two previous torques are equalized, so $\frac{C_f}{L_c} = m_m B_e$ and then $L_c = \frac{C_f}{m_m B_e} = \frac{5.1 \times 10^{-7}}{0.33 \times 47 \times 10^{-6}} = 3.2$ cm. The ratio between the torque at 34cm and the critical torque at 3.2cm is therefore 3.2/34=9%. So the answer is [5 ; 20%[.

B.4.1: Correct expression of $L_c = C_f/(m_m B_e)$.	0.1
B.4.2: L_c value (with units) in [2.0 ; 6.0] cm.	0.1
B.4.3: Correct range: [5%,20%[.	0.1

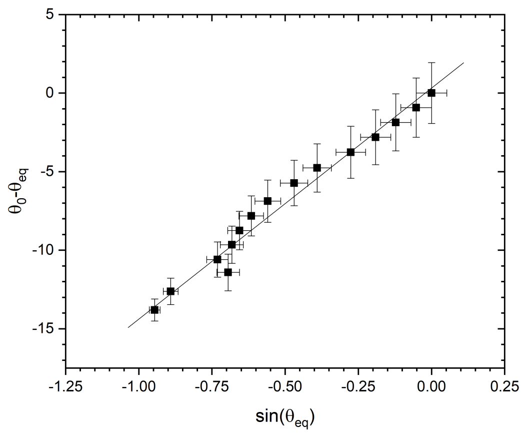
Static regime measurement

We now propose a static measurement of the Earth's magnetic field. Reinsert the magnet into the pod. Use piece (f2a) in Fig. 3 to adjust the angular position θ_0, causing the wire to twist.

B.5	Still at a fixed length of $L = 34\,\text{cm}$, draw an appropriate plot to study how the equilibrium position of the magnet θ_{eq} depends on the angle θ_0, and determine a second value of B_{e}, with its uncertainty.	1.1pt

SOLUTION:

According to the equation of motion in an equilibrium situation, we have : $\dfrac{C_f(\theta_{eq}-\theta_0)}{L} = -m_m B_e \sin(\theta_{eq})$. The protocol therefore involves plotting, for a fixed length L, $\theta_{eq}-\theta_0$ as a function of $\sin(\theta_{eq})$ and checking that it is indeed a linear function, the slope of which $p = \dfrac{m_m B_e L}{C_f}$ can be calculated.



The slope obtained from the measurements is 11.1 ± 1.7 rad^{-1} for $L = 0.34$ m. The Earth magnetic field is then $B_e = \dfrac{pC_f}{m_m L} = \dfrac{11.1 \times 5.1 \times 10^{-7}}{0.33 \times 0.34} = 50.5 \pm 8\,\mu$T. Note that uncertainty is relatively high.

B.5.1: Plot (axes, units).	0.1
B.5.2: 5 measures or more.	0.1
B.5.3: 7 measures or more.	0.1
B.5.4: Identification and calculation of the relevant quantity. Either $(\theta_{eq}-\theta_0) = f(\sin(\theta_{eq}-\theta_e))$ or inverse.	0.2
B.5.5: Experimental plot showing linearity with correct sampling.	0.2
B.5.6: B_e value (with units) in [1.0 ; 10] 10^{-5} T.	0.1
B.5.7: B_e value (with units) in [1.5 ; 7] 10^{-5} T.	0.1
B.5.8. δB_e value (with units) in [0.1 ; 2] 10^{-5} T.	0.1
B.5.9. δB_e value (with units) in [0.6 ; 1] 10^{-5} T.	0.1

B.6	Vary the length L and repeat the previous study for two other lengths to verify the L dependence of the wire torque. Using a final graph that summarizes all the dependencies, determine a new value for B_e, with its uncertainty.	2.3pt

SOLUTION:

Now the length L is varied, for $L = 26$, 12 and 8 cm.

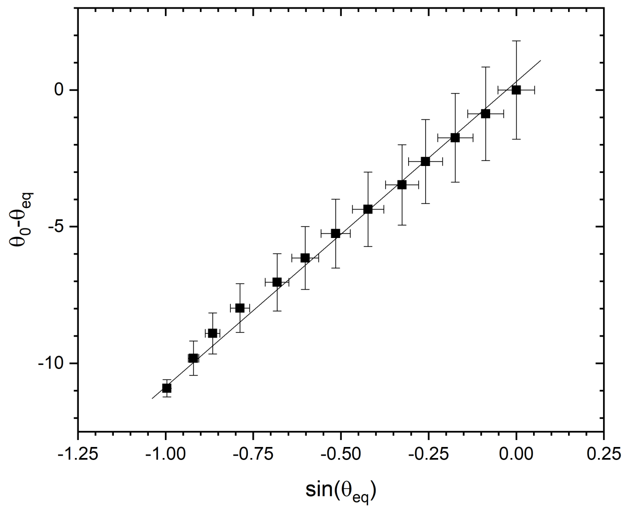


Figure B.6A: $L = 26$ cm.

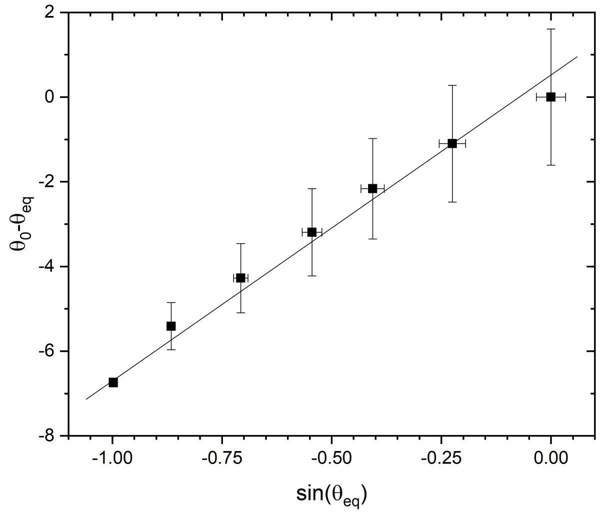


Figure B.6B: $L = 12$ cm.

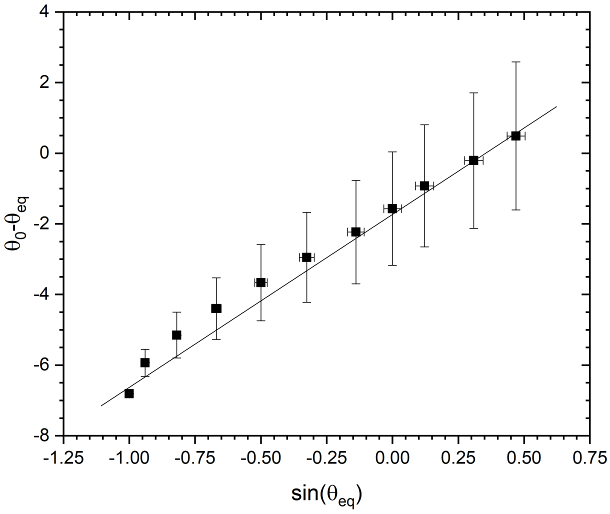


Figure B.6C: $L = 8$ cm.

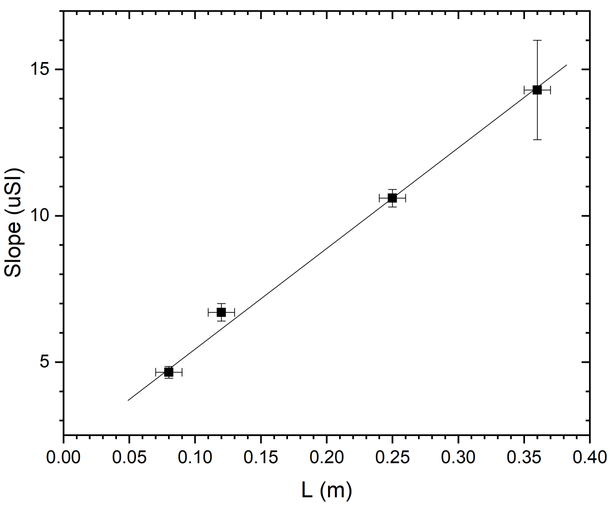
The slopes are

$L = 25$ cm $\quad p = 10.6 \pm 0.3$ rad^{-1},

$L = 12$ cm $\quad p = 6.7 \pm 0.3$ rad^{-1},

$L = 8$ cm $\quad p = 4.6 \pm 0.2$ rad^{-1}.

So we can plot the slopes versus L as



Slopes versus L.

As expected, the slope of figure B.6D is found to be proportional to L, with a slope $p' = 27.1 \pm 1.5$ rad$^{-1}\cdot$m^{-1}. A new value of B_e is deduced from $B_e = \dfrac{p'C_f}{m_m} = \dfrac{5.1 \times 10^{-7} \times 27.1}{0.305} \approx 45.3\ \mu$T. Relative uncertainty is given by

$\frac{\delta p'}{p'} + \frac{\delta C}{C} + \frac{\delta m}{m} = 0.12$, so

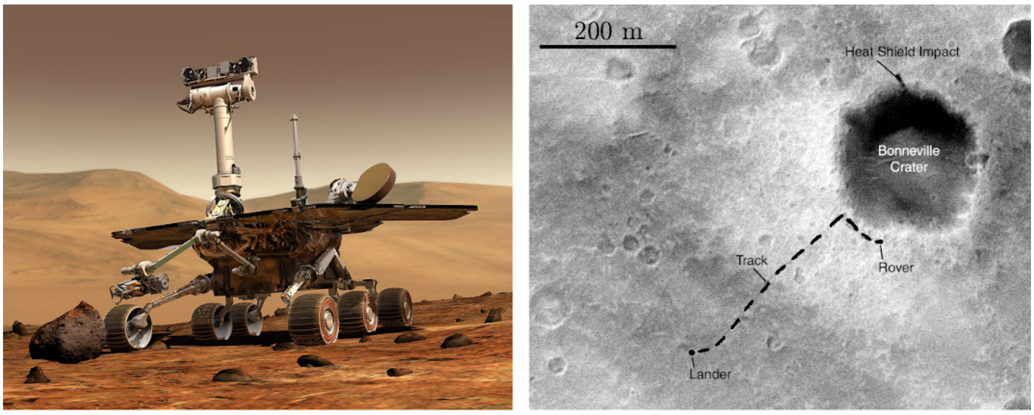
$B_e = 45.3 \pm 5\ \mu$T.

B.6.1. Equilibrium 2 : 5 measures or more	0,1
B.6.2. 7 measures or more	0,1
B.6.3. Calculation of $\theta_{eq} - \theta_0 = f(\sin(\theta_{eq} - \theta_e))$ or inv.	0,1
B.6.4. Plot (axes, units)	0,1
B.6.5. Plot showing linearity with correct sampling	0,2
B.6.6. Slope for L2	0,1
B.6.7. Equilibrium 3 : 5 measures or more	0,1
B.6.8. 7 measures or more	0,1
B.6.9. Calculation of $\theta_{eq} - \theta_0 = f(\sin(\theta_{eq} - \theta_e))$ or inv.	0,1
B.6.10. plot (axes, units)	0,1
B.6.11. Plot showing linearity with correct sampling	0,2
B.6.12. Slope for L3	0,1
B.6.13. Identification and calculation of slope versus L	0,3
B.6.14. Plot (axes, units)	0,2
B.6.15. $B_e = \frac{p' C_f}{m_m}$	0,1
B.6.16. $B_e \in [1.5; 7]10^{-5}$T	0,2
B.6.17. $\delta B_e \in [0.2; 0.8].10^{-5}$T	0,1

Another possible solution is to represent all the measurements at different L in the same graph, by plotting $(\theta_{\text{eq}} - \theta_0)$ versus $L \cdot \sin(\theta_{\text{eq}} - \theta_{\text{e}})$ and measure slope from there. Doing so should allow students to earn the maximum number of points according to the grading scheme.

Sand craters and dunes (10.0 points)

NASA's Spirit rover (**Fig. 1.**(a)) landed on Mars in 2004 to study its geology and potential presence of water. The landing site (**Fig. 1.**(b)) is surrounded by craters of various sizes and sand dunes. During exploration, the rover must avoid getting stuck in the sand dunes of Mars.



(a) (b)

Fig. 1. (a) Artist's view of Spirit. (b) Landing site of the rover on Mars. The scale bar represents $200\,\mathrm{m}$.

The problem has two independent parts A (crater formation) and B (sand trapping) that can be treated in any order. The list of equipment is given below and illustrated in **Fig. 2.**

- (a) Plastic box, needs to be emptied. The empty box will be used to collect the overflowing sand during experiments.
- (b) Bowl.
- (c) Bottle of sand.
- (d) 6 steel balls in a container. The balls have 4 different diameters. The three smallest ones are identical.
- (e) Tape measure.
- (f) Holding device consisting of a wooden tray with rubber feet (f1), a vertical rod (f4), clamping screw (f2) and horizontal rod (f3). The different elements must be assembled as shown in the photo (f).
- (g) Sieve, used to find the small ball if it gets lost in the sand.
- (h) Aluminium rail, $1\,\mathrm{m}$ long.
- (i) Brush to clean the rail and balls of sand if necessary.
- (j) Wooden track.
- (k) Chronometer.
- (l) Adhesive putty.
- (m) Funnel to help to put the sand back into the box at the end.
- (n) Spoon.
- (o) Ruler.

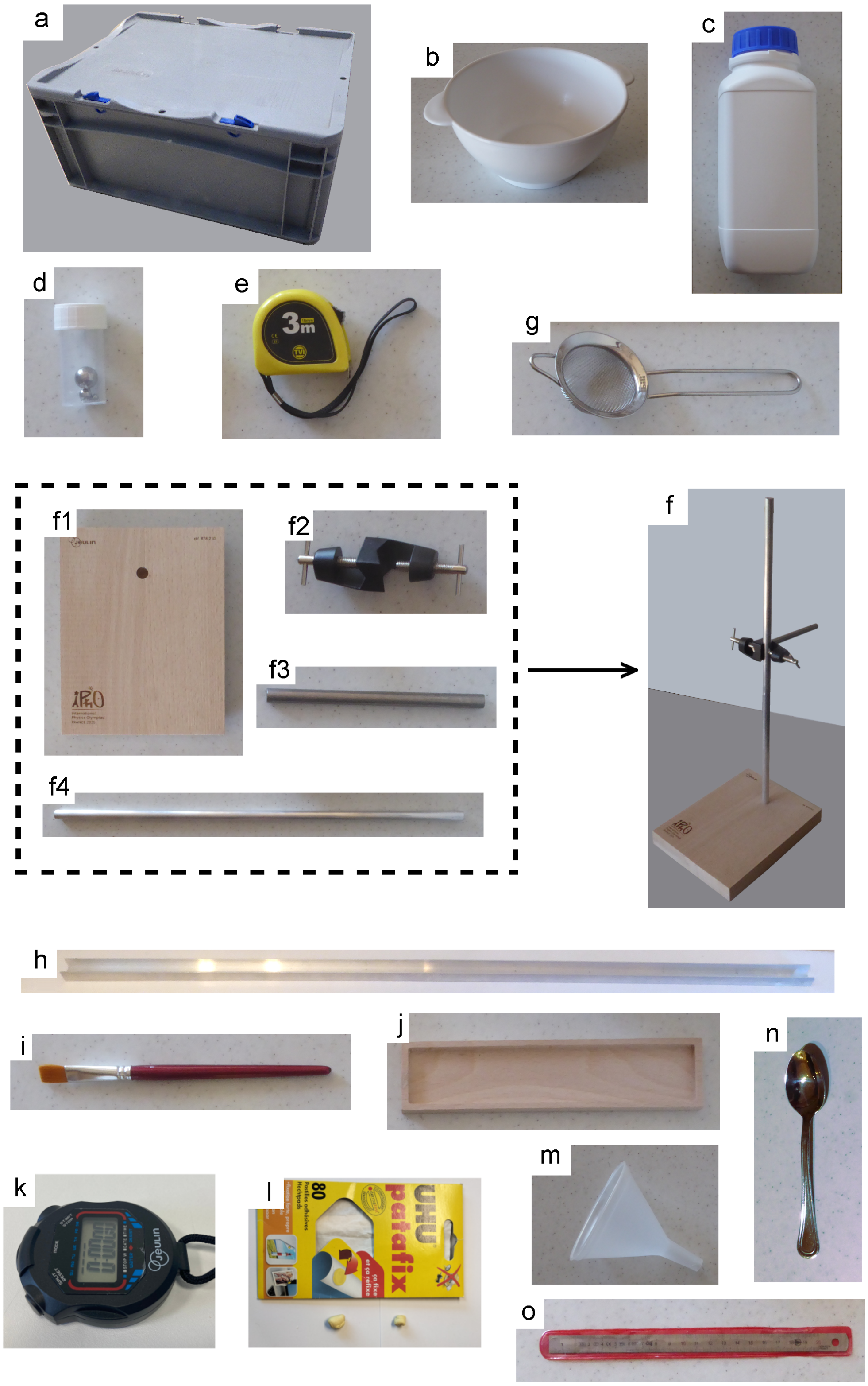


Fig. 2. Photographs of all equipment.

A. Impact craters

Craters on Mars, whose diameter D varies from about $10\,\mathrm{m}$ to several hundreds of km, result from the impact of meteorites. Different models predict how D depends on the impact parameters: impactor diameter d, energy E (**Fig. 3**).

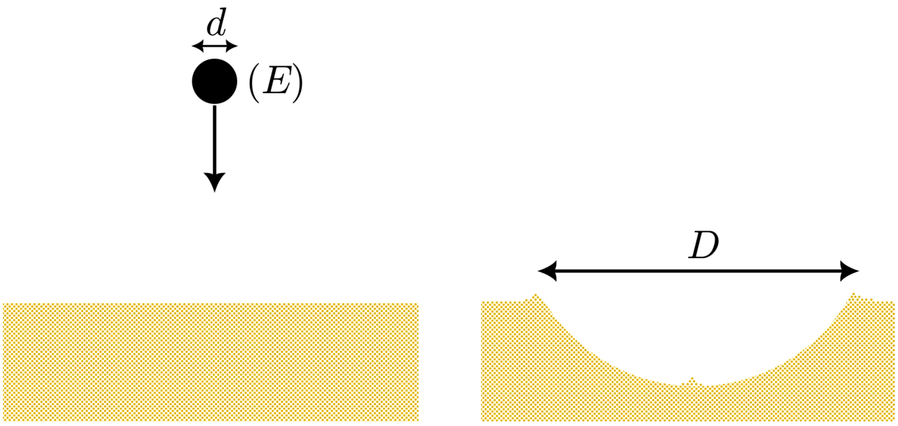


Fig. 3. Crater formation.

Model 1: D depends only on the impactor diameter d

$$D = c_1 d, \tag{1}$$

where c_1 is a dimensionless number independent of E and d.

Model 2: the meteorite energy E is converted through volumic processes during the impact. This model predicts that D is proportional to $E^{1/3}$

$$D = c_2 E^{1/3} \tag{2}$$

where c_2 is a parameter independent of E and d.

Model 3: E is used to eject material outside the crater. Under this assumption

$$D = c_3 E^{1/4} \tag{3}$$

where c_3 is a parameter independent of E and d.

Here, we perform experiments on crater formation at a centimeter scale to compare the three models. Steel balls of different diameters d and masses m, with a density $\rho_a = 7.8 \times 10^3\,\mathrm{kg \cdot m^{-3}}$ (item (**d**) of the equipment list), act as the meteorites.

Ball #1	$d_1 = 2.0\,\mathrm{mm}$	$m_1 = 0.033\,\mathrm{g}$
Ball #2	$d_2 = 5.0\,\mathrm{mm}$	$m_2 = 0.51\,\mathrm{g}$
Ball #3	$d_3 = 9.0\,\mathrm{mm}$	$m_3 = 3.0\,\mathrm{g}$
Ball #4	$d_4 = 16.0\,\mathrm{mm}$	$m_4 = 17\,\mathrm{g}$

The bowl (**b**) filled with sand (**c**) is placed inside the emptied plastic box (**a**) that will help collect the excess sand. The bowl is filled completely with sand and the surface is carefully leveled with the edge of the ruler (**o**). Avoid compacting the sand! To release the ball above the bowl, one can use the stand equipped with a rod and thumbscrew (**f**). The rod serves as a guide to release the ball directly above the bowl and also to measure the drop height h above the surface, which will be measured using the tape measure (**e**).

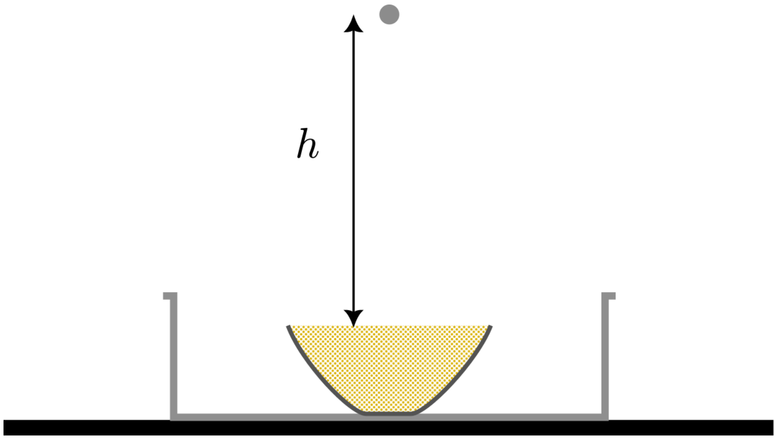


Fig. 4. Crater formation experimental setup.

Drop ball #3 from a height $h = 50\,\text{cm}$ and measure the diameter D of the crater formed. Repeat the experiment 5 times. After each impact, mix the sand with the spoon (**n**), and level it carefully with the edge of ruler (**o**). Avoid compacting the sand! If needed, use the sieve (**g**) to find the ball if it gets lost in the sand.

A.1	Present your results in a table and give D with its uncertainty.	0.6pt

SOLUTION:

D(mm)	23	24	22	25	25

D=(23.8±1.2)mm

A1(1) : 2 measures of D between 22mm and 26mm	0.2pt
A1(2) : 3 more measures of D between 22mm and 26mm	0.2pt
A1(3) : mean value of D between 23mm and 25mm	0.1pt
A1(4) : uncertainty on D between 0.5 mm and 2mm	0.1pt

During the fall, the air drag force is

$$F = \frac{1}{8}\pi d^2 \rho_0 C_x v^2 \tag{4}$$

where v is the ball velocity, $\rho_0 \simeq 1.2\,\text{kg}\cdot\text{m}^{-3}$ is the air density and C_x is a dimensionless coefficient of order unity.

The air drag force is negligible if the ball is dropped from a height less than the maximum drop height $h_{\max}$, defined as the height at which the air drag force remains less than 10 % of the weight throughout the fall.

A.2	Determine the theoretical expression for the maximum drop height $h_{\max}$. Calculate $h_{\max}$ numerically for the four available balls.	0.5pt

SOLUTION:

If the friction with air in neglected, the maximum speed writes $v_{\max} = \sqrt{2gh}$ and the corresponding air friction is $F = \frac{1}{8}\pi d^2 \rho_0 C_x.(2gh)$. If we want $F < mg/10$ then we obtain

$$h < 0.1\frac{2}{3}\frac{\rho_a}{\rho_0}\frac{1}{C_x}d$$

A2(1) : $h_{\max} = 0.1\frac{2}{3}\frac{\rho_a}{\rho_0}\frac{1}{C_x}d$ or any equivalent formula involving other variables.	0.4pt
A2(2) : 4 values for hmax = (0.9m ; 2m ; 4m ; 7m)	0.1pt

Investigate the relationship between D and E experimentally in order to compare the three power laws presented in the introduction. Find out if the exponent changes across the range of energies tested. To achieve this, take a series of measurements by dropping the balls from different heights. A wide range of energies must be covered. The balls can be dropped from heights of up to $h = 2\,\mathrm{m}$ in order to reach high values of E while respecting the condition established in **A.2.** For each set of parameters, repeat the experiment only twice, and compute the mean value D.

A.3	Present your results in a table: mass of the ball m, drop height h, impact energy E, crater diameter D.	1.7pt

SOLUTION:

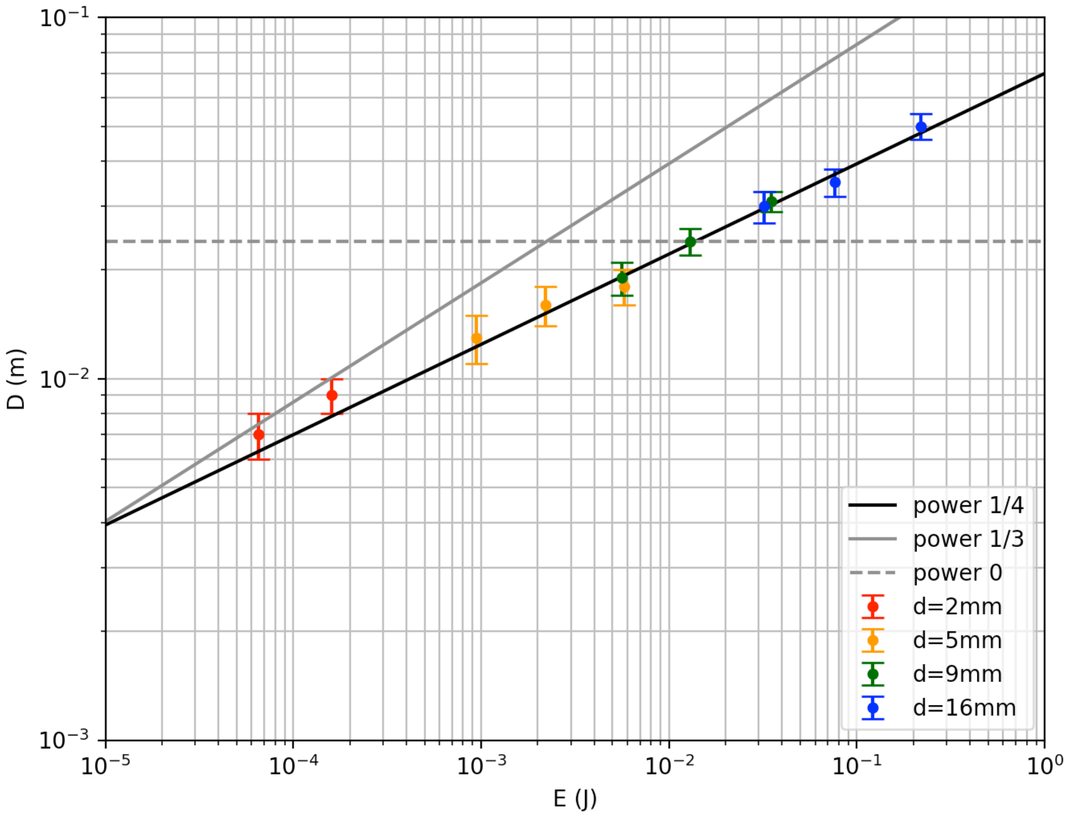
In order to cover a wide range of impact energies, we will release the small ball from a low height (h=10cm) and the big ball from height up to 2m (no need to drop the small ball from high). A key point is to reform the sand after each impact. If not, the sand becomes harder and the craters will be smaller. The energy E=mgh varies from 3E-5 J (ball #1, h=10cm) up to 0.4J (ball #4, h=2m).

The expected values established by pre-IPhO experiments follow $D = 6.92(mh)^{0.25}$ where D is in mm, m is in g and h in cm.

A3(1) : N>5,5 correct values of D (|D-Dref|<0.15*Dref)	0.3pt
A3(2) : N>8,5 correct values of D	0.3pt
A3(3) : N>11,5 correct values of D	0.3pt
A3(4) : Correct calculation of E=mgh (|E-Eth|<0,1*Eth)	0.2pt
A3(5) : 2 decades for E (with 2 points/decade)	0.2pt
A3(6) : 3 decades for E (with 2 points/decade)	0.2pt
A3(7) : more than 3,5 decades for E	0.2pt

A.4	Plot your results on the graph paper of your choice (logarithmic or linear). On the graph representation, add lines corresponding to models 1, 2 and 3. State which of the three theoretical models best fits the experimental data.	1.2pt

SOLUTION:



A4(1) : axes with label and units	0.1pt
A4(2) : theoretical straight lines of slope 1/3 and 1/4 (anywhere)	0.2pt
A4(3) : theoretical straight horizontal line (exponent 0)	0.1pt
A4(4) : N>4 points on the graph	0.2pt
A4(5) : two points of the graph in coherence with the values in A3	0.2pt
A4(6) : points form a straight line	0.2pt
A4(7) : slope mesured and conclusion 1/4	0.2pt

B. Rolling and bogging in sand

Five years after landing, the rover Spirit bogs in the sands of a Martian dune for good. Rolling in sand is particularly delicate as the motion of grains dissipates a lot of energy. Here, we study the braking of a ball rolling in sand. The ball, initially at rest, is first accelerated on a rail inclined at an angle θ, then slowed down on a bed of sand.

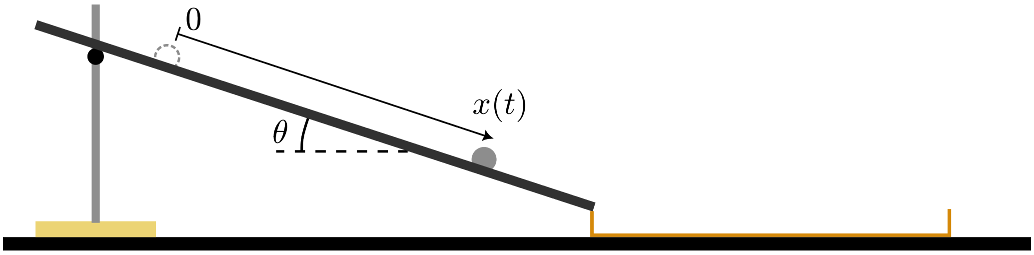


Fig. 5. Inclined rail (**h**) combined with the wooden track (**j**).

Ball motion along the rail

Ball #4 is released with no initial speed from an arbitrary point on the rail (**h**), chosen as the origin of the x-axis ($x = 0$) (**Fig. 5**). Let $x(t)$ denote the position of the ball along the rail. The moment of inertia of a ball of mass m and diameter d with respect to an axis passing through it center is given by $J = md^2/10$. The kinetic energy K of a ball moving at speed v while rotating at angular speed ω is

$$K = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2. \tag{5}$$

We assume that the ball rolls on the rail without slipping and neglect any energy dissipation.

B.1	Express the position x of the ball as a function of time t, angle θ and acceleration of gravity g.	0.4pt

SOLUTION:

Energy theorem (no dissipation) together with the kinematic relation $v = \omega R$ give a rapid answer. $dK/dt = -mgv\sin\theta$ leads to $x(t) = \frac{1}{1+\frac{4J}{md^2}}.\frac{1}{2}g\sin\theta t^2$. Because of rolling, the ball is 5/7 slower than an hypothetic material point.

B1(1) : $x(t) = \frac{1}{1+\frac{4J}{md^2}}\frac{1}{2}g\sin\theta t^2 = \frac{5}{7}\frac{1}{2}g\sin\theta t^2$	0.4pt

One end of the rail (**h**) rests on the edge of the wooden track (**j**), which is at this point empty of sand. The other end of the rail is supported by the stand (**f**) in such a way that it forms an angle of inclination $\theta = 5°$ with the horizontal. Make sure to perform this adjustment carefully. The rail is secured in place (on both sides) using adhesive putty (**l**).

Use a chronometer (**k**) to measure the time t_{50} taken by the ball to travel a distance $l = 50\,\mathrm{cm}$ along the rail.

B.2	Take 5 measurements and present the result along with the order of magnitude of its statistical uncertainty.	0.7pt

SOLUTION:

t_{50} (s)	1.28	1.35	1.39	1.32	1.33

$$t_{50} = (1.33 \pm 0.04)s$$

B2(1) : 1mesure of t_{50} between 1.2s and 1.4s	0.2pt
B2(2) : 4 more mesures of t_{50} between 1.2s and 1.4s	0.2pt
B2(3) : mean value of t_{50} between 1.25s and 1.35s	0.2pt
B2(4) : uncertainty of t_{50} between 0.02s and 0.30s	0.1pt

Measure t with the order of magnitude of its statistical uncertainty for at least 8 different values of ℓ.

B.3	Present your results in a table.	0.8pt

SOLUTION:

Since we can anticipate the arrival of the ball at the bottom of the rail, the measurements are quite reproducible.

ℓ(cm)	10	20	30	40	50
t(s)	0.54±0.03	0.87±0.05	1.04±0.05	1.16±0.06	1.33±0.04

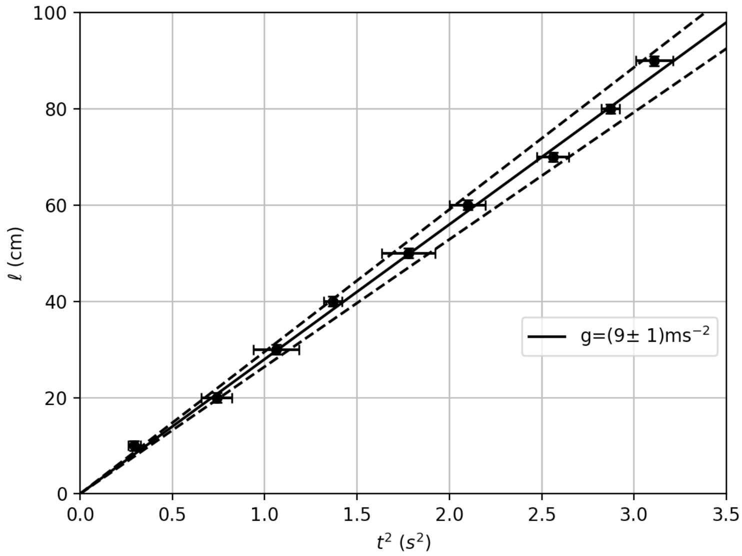
ℓ(cm)	60	70	80	90	100
t(s)	1.45±0.04	1.60±0.07	1.71±0.06	1.78±0.05	1.83±0.06

B3(1) : measures of t with uncertainty for 4 different values of ℓ ($\lvert t - t_{th}\rvert < 0.1 \times t_{th}$)	0.3pt
B3(2) : 4 more measures of t	0.3pt
B3(3) : ℓ goes from 10cm up to 90cm	0.2pt

B.4	Plot your results with error bars to confirm the law established at question **B.1**. Deduce an experimental estimate of the constant g with its uncertainty.	1pt

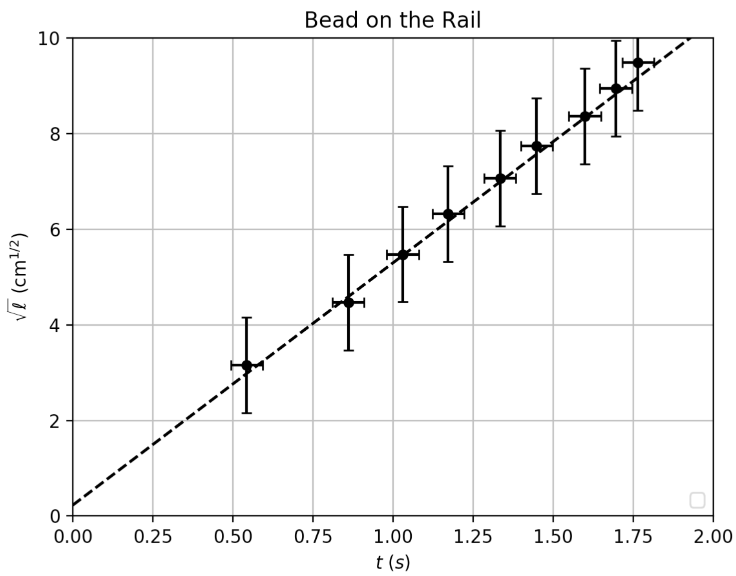
SOLUTION:

We can plot ℓ as a function of t^2 or vice versa to obtain a straight line.



The slope is $\frac{5}{14} g \sin\theta$. We find g=(9$\pm$1)ms^{-2}. This value is very sensitive to an error on the slope of the rail. An error of 1° (out of 5°) leads to an error of 2ms^{-2} on the value of g

One can also plot $\sqrt{\ell}$ as a function of t to detect a systematic shift error on t.



B4(1) : smart choice for the axes (ℓ, t^2) or anything that ends to a line	0.2pt
B4(2) : Presence of error bars for t	0.2pt
B4(3) : Adjustment made by a straight line.	0.2pt
B4(4) : value of g between 6ms^{-2} and 14 ms^{-2}	0.2pt
B4(5) : values of g between 8ms^{-2} and 12 ms^{-2}	0.1pt
B4(6) : Uncertainty on g of the order of 1 ms^{-2}	0.1pt

Motion of the ball in sand

We note ℓ the distance travelled by the ball on the rail. On the sand, the ball comes to a stop after travelling a distance L as defined in **Fig. 6**.

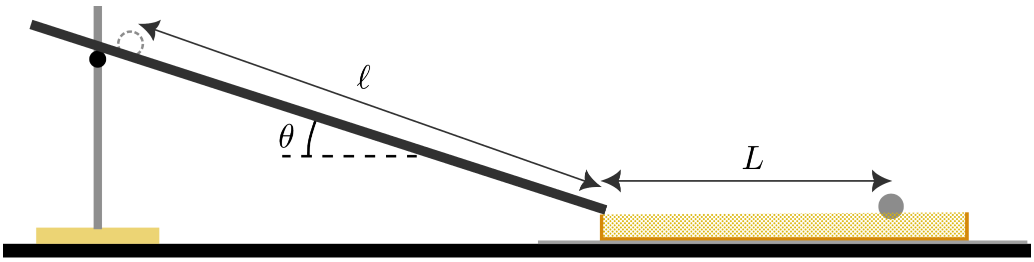


Fig. 6. Acceleration over a distance ℓ and stopping over a distance L.

It is thus slowed down by a drag force T which may have two possible origins:

- **Model #1 (solid friction):** as between two solids in relative motion, the sand exerts on the ball a constant drag force $T = -\mu_{\text{eff}} mg$, where μ_{eff} is the effective drag coefficient of the ball-sand contact and m is the mass of the ball.
- **Model #2 (fluid drag):** the drag force depends linearly on the ball velocity, $T = -kv$ where k is a constant and v the norm of the velocity.

The goal here is to determine which proposition best describes the observed braking behavior.

When moving in sand, the ball is modelled as a point mass. Given the small value of the slope of the rail, we neglect any energy loss in the transition between the rail and the sand track. Establish the theoretical law linking L to ℓ in each of the two situations (solid friction or fluid drag). The two suggestions lead to a power law of the form $L \sim \ell^{\alpha}$ in which the exponent α takes two different values.

B.5	For model 1 and model 2, give the relationship between L and ℓ and the value of α.	0.6pt

SOLUTION:

Energy considerations for model #1 are straightforward : ~~$mgh = \mu_{\text{eff}} mgL$ wich lead to $L = (\sin\theta/\mu_{\text{eff}})\ell$~~ $\frac{1}{2}mv_0^2 = \mu_{\text{eff}} mgL$ where v_0 is the initial velocity in sand. This leads to $L = \frac{5}{7}(\sin\theta/\mu_{\text{eff}})\ell$ and then $\alpha = 1$.

Some more calculus are needed for suggestion #2 : we must solve the differential equation for v(t) and we find $v(t) = v_0 \exp(-t/\tau)$ where $\tau = m/k$ and ~~$v_0 = \frac{5}{7}\sqrt{2g\ell\sin\theta}$~~ $v_0 = \sqrt{\frac{5}{7}}\sqrt{2g\ell\sin\theta}$. An integration gives

$x(t) = v_0\tau(1-\exp(-t/\tau))$ with tends to

~~$L = \frac{5}{7}\frac{m}{k}\sqrt{2g\sin\theta}\sqrt{\ell}$~~ $L = \sqrt{\frac{5}{7}}\frac{m}{k}\sqrt{2g\sin\theta}\sqrt{\ell}$ and $\alpha = 1/2$.

B5(1) : model #1: $L = \frac{5}{7}(\sin\theta/\mu_{\text{eff}})\ell$. The answer $L = (\sin\theta/\mu_{\text{eff}})\ell$ will also be accepted.	0.1pt
B5(2) : model #1: $\alpha = 1$	0.1pt
B5(3) : model #2: ~~$L = \frac{5}{7}\frac{m}{k}\sqrt{2g\sin\theta}\sqrt{\ell}$~~ $L = \frac{m}{k}\sqrt{\frac{5}{7}}\sqrt{2g\sin\theta}\sqrt{\ell}$. The answer $L = \frac{m}{k}\sqrt{2g\sin\theta}\sqrt{\ell}$ will also be accepted.	0.3pt
B5(4) : model #2: $\alpha = 1/2$	0.1pt

Place the wooden track (**j**) on a sheet of paper. Fill the track with sand and prepare a uniform layer by carefully scraping the surface with the ruler. Avoid compacting the sand! Adjust again carefully the angle of the rail to $\theta = 5°$. Release ball #4 ($d_4 = 16.0\,\text{mm}$) on the inclined rail so that the distance travelled on the rail is $l = 50\,\text{cm}$.

*Before each run, stir the sand, refill the track and scrape the surface again. Clean the rail and the ball from sand by using the brush (**i**). At the end of the experiment, use the sheet of paper as a funnel to put the sand in excess back in the bottle.*

B.6	Measure the distance L_{50} travelled in the sand until the ball comes to a stop. Perform several measurements (at least 5) to determine L_{50} along with its unit and uncertainty.	0.8pt

SOLUTION:

If the layer is not carefully redone after each run, the sand will get harder and the ball will run out of the track.

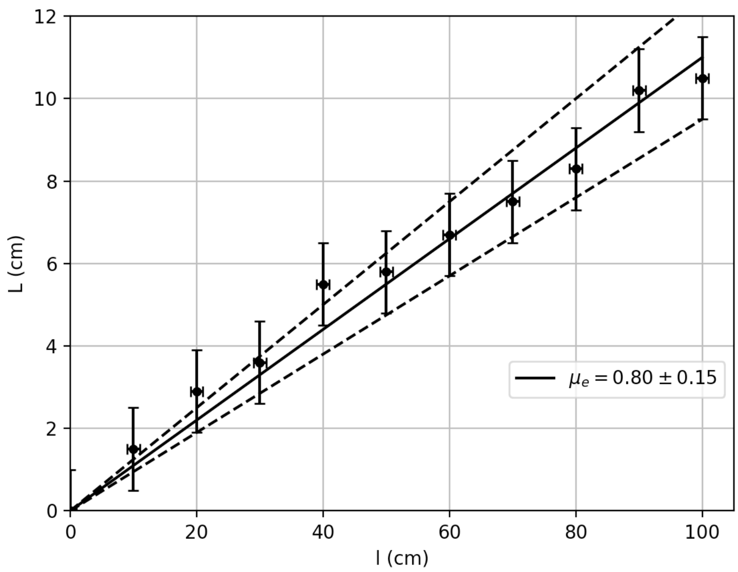
L_{50} (cm)	7.0	6.7	5.8	6.7	6.1

$L_{50} = (6.4 \pm 0.5)$ cm

B6(1) : 3 measures of L_{50} between 5,5cm and 7,5cm.	0.4pt
B6(2) : 2 more measures between 5,5cm and 7,5cm	0.2pt
B6(3) : mean value of L_{50} between 5.8cm and 7.2cm	0.1pt
B6(4) : uncertainty on L_{50} between 0.2cm and 1.0cm	0.1pt

B.7	After several measurements for at least 8 values of ℓ (keeping $\theta = 5°$), plot L with its error bars as a function of ℓ and conclude which model best describes the drag force T.	1.5pt

SOLUTION:



The point are compatible with a straight line and a "solid-like" friction model (suggestion #1)

B7(1) : 4 measures of L for different ℓ	0.3pt
B7(2) : 4 more measures of L	0.2pt
B7(3) : ℓ varies (at least) from 10cm up to 90cm	0.1pt
B7(4) : graph L as a function of ℓ or log(L) as a function of log(ℓ), axes, values and units	0.1pt
B7(5) : more than 2,5 points on the graph	0.1pt
B7(6) : more than 5,5 points on the graph	0.2pt
B7(7) : error bars between $\pm$0.5cm and $\pm$1cm for L	0.1pt
B7(8) : a linear law is ploted, and compatible with the points.	0.2pt
B7(9) : conclusion $\alpha = 1$ and "solid-like" friction in sand.	0.2pt

B.8	Based on the chosen model, specify the value of the coefficient μ_{eff} or k that characterizes the force T.	0.2pt

SOLUTION:

The relation is $L = (\sin\theta / \mu_{\text{eff}})\ell$

$$\mu_{\text{eff}} = 0.8 \pm 0.1$$

B8(1) : μ_{eff} between 0.6 and 1.0	0.2pt