A. Earth as a blackbody

A-1. All the energy emitted from the surface of the Sun, will reach a sphere of radius d , therefore:

$$\sigma T_S^4.(4\pi R_S^2) = (4\pi d^2).S_0$$

$$S_0 = \sigma T_S^4.\left(\frac{R_S}{d}\right)^2 = 5.67\times10^{-8}\frac{\text{W}}{\text{m}^2\text{K}^4}\times(5.77\times10^3\ \text{K})^4\times\left(\frac{6.96\times10^8\ \text{m}}{1.5\times10^{11}\ \text{m}}\right)^2 = 1.35\times10^3\frac{\text{W}}{\text{m}^2}$$

A-1 (0.6 pt)	
$S_0 = \sigma T_S^4.\left(\frac{R_S}{d}\right)^2$	, Numerical value of $S_0 = 1.35\times10^3\ \text{W/m}^2$

A-2. It is assumed that the Earth is in thermal equilibrium. Therefore, the energy it receives per unit time should be equal to the energy it radiates per unit time. The Earth's cross-section intercepting the solar radiation at this distance has an area of πR_E^2 , but the Earth radiates heat from all points on its surface with an area of $4\pi R_E^2$, so:

$$\pi R_E^2.S_0 = 4\pi R_E^2\sigma T_E^4 \quad \rightarrow T_E = \left(\frac{S_0}{4\sigma}\right)^{\frac{1}{4}} = 278\ \text{K}$$

A-2 (0.6 pt)	
$T_E = \left(\frac{S_0}{4\sigma}\right)^{\frac{1}{4}} = \sqrt{\frac{R_S}{2d}}T_S$	, Numerical value of $T_E = 278\ \text{K}$

A-3. The radiation is maximum at the wavelength for which the derivative of u with respect to λ is zero:

$$\frac{du}{d\lambda} = \frac{2\pi hc^2}{\lambda^6}.\frac{1}{exp(\frac{hc}{\lambda k_B T})-1}.\left[-5+\frac{hc}{\lambda k_B T}\frac{exp(\frac{hc}{\lambda k_B T})}{exp(\frac{hc}{\lambda k_B T})-1}\right]$$

$$\frac{du}{d\lambda}|_{\lambda=\lambda_m} = 0 \quad \Rightarrow \quad \frac{hc}{\lambda_m k_B T}\frac{exp\left(\frac{hc}{\lambda_m k_B T}\right)}{exp\left(\frac{hc}{\lambda_m K_B T}\right)-1} = 5$$

Defining $x_m \equiv \frac{hc}{\lambda_m k_B T}$ we obtain the following transcendental equation:

$$5(1-e^{-x_m}) - x_m = 0$$

A-3 (0.4 pt)
$f(x) = 5(1 - e^{-x}) - x$

A-4. The first guess is $x_{\mathrm{m}}^{(1)} = 5$. Substituting repeatedly for x_{m} we can continue as follows:

$$x_{\mathrm{m}}^{(2)} = 5(1 - e^{-5}) = 4.97$$
$$x_{\mathrm{m}}^{(3)} = 5(1 - e^{-4.97}) = 4.97$$

Further iterations do not change the value of x_{m} to three significant figures, so:

$$\lambda_{\mathrm{m}} T = \frac{hc}{x_{\mathrm{m}} k_{\mathrm{B}}} = b = 1240\ \mathrm{eV \cdot nm}\ \times \frac{1}{4.97 \times 8.62 \times 10^{-5}\ \mathrm{eVK^{-1}}} = 2.89 \times 10^{6}\ \mathrm{nm \cdot K}$$

A-4 (0.4 pt)	
$x_{\mathrm{m}} = \{4.96, 4.97\}$	, Numerical value of $b =\ [2.89, 2.90] \times 10^{6}\ \mathrm{nm \cdot K}$

A-5. Using Wien's displacement law and the constant b obtained in the previous part, we can calculate the wavelength at which the radiation from the Sun and the Earth reaches its maximum:

$$\lambda_{\mathrm{max}}^{\mathrm{Sun}} = \frac{b}{T_{\mathrm{S}}} = \frac{2.89 \times 10^{6}\ \mathrm{nm \cdot K}}{5.77 \times 10^{3}\ \mathrm{K}} = [5.01, 5.02] \times 10^{2}\ \mathrm{nm}$$

$$\lambda_{\mathrm{max}}^{\mathrm{Earth}} = \frac{b}{T_{\mathrm{E}}} = \frac{2.89 \times 10^{6}\ \mathrm{nm \cdot K}}{278\ \mathrm{K}} = 1.04 \times 10^{4}\ \mathrm{nm}$$

A-5 (0.2 pt)	
$\lambda_{\mathrm{max}}^{\mathrm{Sun}} = [5.01, 5.02] \times 10^{2}\ \mathrm{nm}$	, $\lambda_{\mathrm{max}}^{\mathrm{Earth}} = 1.04 \times 10^{4}\ \mathrm{nm}$

A-6. From the diagram, it can clearly be seen that $\gamma \tilde{u}_{\mathrm{S}}(\lambda_{\mathrm{max}}^{\mathrm{S}}) = u(\lambda_{\mathrm{max}}^{\mathrm{Earth}}, T_{\mathrm{E}})$, so we have:

$$\tilde{u}_{\mathrm{S}}(\lambda_{\mathrm{max}}^{\mathrm{Sun}}) = \left(\frac{R_{\mathrm{S}}}{d}\right)^{2} \frac{2\pi hc^{2}}{(\lambda_{\mathrm{max}}^{\mathrm{Sun}})^{5}} \frac{1}{exp\left(\frac{hc}{\lambda_{\mathrm{max}}^{\mathrm{Sun}} k_{\mathrm{B}} T_{\mathrm{S}}}\right) - 1} = \left(\frac{R_{\mathrm{S}}}{d}\right)^{2} \frac{2\pi hc^{2}}{(\lambda_{\mathrm{max}}^{\mathrm{Sun}})^{5}} \frac{1}{exp\left(\frac{hc}{k_{\mathrm{B}} b}\right) - 1}$$

$$u(\lambda_{\mathrm{max}}^{\mathrm{Earth}}, T_{\mathrm{E}}) = \frac{2\pi hc^{2}}{(\lambda_{\mathrm{max}}^{\mathrm{Earth}})^{5}} \frac{1}{exp\left(\frac{hc}{\lambda_{\mathrm{max}}^{\mathrm{Earth}} k_{\mathrm{B}} T_{\mathrm{E}}}\right) - 1} = \frac{2\pi hc^{2}}{(\lambda_{\mathrm{max}}^{\mathrm{Earth}})^{5}} \frac{1}{exp\left(\frac{hc}{k_{\mathrm{B}} b}\right) - 1}$$

Dividing these two quantities we'll find:

$$\gamma = \left(\frac{d}{R_S}\right)^2 \times \left(\frac{T_E}{T_S}\right)^5 = [1.20,1.21] \times 10^{-2}$$

A-6 (0.8 pt)	
$\gamma = \left(\frac{d}{R_S}\right)^2 \times \left(\frac{T_E}{T_S}\right)^5 = \left(\frac{d}{R_S}\right)^2 \times \left(\frac{\lambda_{max}^{Sun}}{\lambda_{max}^{Earth}}\right)^5$	, Numerical value of $\gamma = [1.20,1.21] \times 10^{-2}$

B. The Greenhouse Effect

B-1. Both the Earth and its atmosphere are in thermal equilibrium, so one can write an equation that balances the input and output powers. For the Earth we have:

$$(\pi R_E^2)(1 - r_A)S_0 + (4\pi R_E^2)\sigma T_A^4 = (4\pi R_E^2)\sigma T_E^4,$$

and for the atmosphere:

$$(4\pi R_E^2)\sigma T_E^4 = 2(4\pi R_E^2)\sigma T_A^4.$$

Note that the coefficient 2 on the right-hand side of the equation is due to the atmosphere radiating heat on both sides (above and below). Eliminating T_E from the two relations we obtain:

$$T_A = \left(\frac{(1 - r_A)\frac{S_0}{4}}{\sigma}\right)^{\frac{1}{4}} = 2.58 \times 10^2 \text{ K} \quad \Rightarrow \quad T_E = (2T_A^4)^{\frac{1}{4}} = 3.07 \times 10^2 \text{ K}$$

B-1 (1.0 pt)	
$T_A = \left(\frac{(1-r_A)\frac{S_0}{4}}{\sigma}\right)^{\frac{1}{4}}$	, Numerical value of $T_A = 2.58 \times 10^2$ K
$T_E = \left(\frac{(1-r_A)\frac{S_0}{2}}{\sigma}\right)^{\frac{1}{4}}$	, Numerical value of $T_E = 3.07 \times 10^2$ K

B-2. As can be seen in the figure, a fraction $(1 - r_A)$ of the solar radiation reaches the Earth's surface after traversing the atmosphere. A fraction r_E of this light is reflected back and reaches the atmosphere, where a fraction r_A is reflected and returns to the Earth's surface. This process repeats *ad infinitum* and the sum of the powers transmitted at all these instances, determines the albedo. Denoting the power returned to space after n reflections by $\tilde{S}_n$, we'll have $\tilde{S}_0 = r_A S_0$ and the remaining power i.e. $(1-r_A)S_0$, reaches the Earth's surface. From this power, $(1-r_A)r_E S_0$ is reflected, and a fraction $1-r_A$ of it is transmitted through the atmosphere to the space, hence:

$$\tilde{S}_1 = (1-r_A)^2 r_E S_0 = \frac{(1-r_A)^2}{r_A} r_E \tilde{S}_0$$

The power that is reflected back to the Earth by the atmosphere after $(n-1)$ reflections is $\tilde{S}_{n-1}\left(\frac{r_A}{1-r_A}\right)$, of which a fraction r_E is again sent back towards the atmosphere on the n'th reflection, and the atmosphere allows a fraction $1-r_A$ of this reflected power to escape into the space, thus:

$$\tilde{S}_n = \frac{\tilde{S}_{n-1}}{1-r_A} r_A r_E \times (1-r_A) = r_A r_E \tilde{S}_{n-1} = (r_A r_E)^{n-1} \tilde{S}_1$$

By adding all these terms, one obtains the power returned per unit area from the Earth-atmosphere system:

$$\tilde{S} = \sum_{n=0}^{\infty} \tilde{S}_n = \tilde{S}_0 + \tilde{S}_1 \sum_{n=1}^{\infty} (r_A r_E)^{n-1} = r_A S_0 + (1-r_A)^2 r_E S_0 \times \frac{1}{1-r_A r_E}$$

$$= \left[r_A + \frac{(1-r_A)^2 r_E}{1-r_A r_E}\right] \times S_0$$

Dividing by the solar constant we get the value for albedo:

$$\alpha = \frac{\tilde{S}}{S_0} = r_A + \frac{(1-r_A)^2 r_E}{1-r_A r_E} = 3.13 \times 10^{-1}$$

B-2 (1.6 pt)	
$\alpha = r_A + \frac{(1-r_A)^2 r_E}{1-r_A r_E}$	, Numerical value of $\alpha = 3.13 \times 10^{-1}$

B-3. Again, thermal equilibrium requires the input and output powers to be equal both for the Earth and for the atmosphere, the only difference being that the Earth absorbs now a fraction $1-\alpha$ of the Sun's radiation. Thus, for Earth we have:

$$(4\pi R_E^2)\epsilon\sigma T_A^4 + (\pi R_E^2)(1-\alpha)S_0 = (4\pi R_E^2)\sigma T_E^4,$$

and for the atmosphere:

$$(4\pi R_E^2)\epsilon\sigma T_E^4 = 2(4\pi R_E^2)\epsilon\sigma T_A^4$$

$$T_E = \left[\frac{(1-\alpha)}{2\sigma(2-\epsilon)} S_0\right]^{\frac{1}{4}} \quad , \quad T_A = \left(\frac{T_E^4}{2}\right)^{\frac{1}{4}}$$

$$\epsilon = \frac{\left[\sigma T_{\mathrm{E}}^4 - \frac{(1-\alpha)}{4} S_0\right]}{\sigma T_{\mathrm{A}}^4} = 2\frac{\left[\sigma T_{\mathrm{E}}^4 - \frac{(1-\alpha)}{4} S_0\right]}{\sigma T_{\mathrm{E}}^4} = [8.07, 8.11] \times 10^{-1}$$

B-3 (1.0 pt)	
$T_{\mathrm{E}} = \left[\frac{(1-\alpha)}{2\sigma(2-\epsilon)} S_0\right]^{\frac{1}{4}}$	, Numerical value of $\epsilon = [8.07\,, 8.11] \times 10^{-1}$

B-4.

$$\frac{dT_{\mathrm{E}}}{d\epsilon} = \frac{1}{4}\left[\frac{(1-\alpha)S_0}{2\sigma(2-\epsilon)}\right]^{\frac{1}{4}} \frac{1}{(2-\epsilon)}$$

$$dT_{\mathrm{E}} = \frac{dT_{\mathrm{E}}}{d\epsilon}\epsilon\frac{d\epsilon}{\epsilon} = \left[\frac{4\sigma T_{\mathrm{E}}^4}{(1-\alpha)S_0} - 1\right]\frac{T_{\mathrm{E}}}{4} \times 0.01 = [4.87, 4.92] \times 10^{-1}$$

B-4 (0.8pt)	
$\frac{dT_{\mathrm{E}}}{d\epsilon} = \frac{1}{4}\left[\frac{(1-\alpha)S_0}{2\sigma(2-\epsilon)}\right]^{\frac{1}{4}} \frac{1}{(2-\epsilon)}$	, Numerical value of $\delta T_{\mathrm{E}} = [4.87, 4.92] \times 10^{-1}$ K

B-5. The equations for thermal equilibrium are similar to those for Part B.3, only a non-radiative thermal current needs to be added. For the Earth:

$$(\pi R_{\mathrm{E}}^2)(1-\alpha)S_0 + (4\pi R_{\mathrm{E}}^2)\epsilon\sigma T_{\mathrm{A}}^4 = (4\pi R_{\mathrm{E}}^2)\sigma T_{\mathrm{E}}^4 + (4\pi R_{\mathrm{E}}^2)k(T_{\mathrm{E}} - T_{\mathrm{A}}),$$

and for the atmosphere:

$$(4\pi R_{\mathrm{E}}^2)\epsilon\sigma T_{\mathrm{E}}^4 + (4\pi R_{\mathrm{E}}^2)k(T_{\mathrm{E}} - T_{\mathrm{A}}) = 2(4\pi R_{\mathrm{E}}^2)\epsilon\sigma T_{\mathrm{A}}^4.$$

After completing the calculations, we will have:

$$\epsilon = \frac{\sigma T_{\mathrm{E}}^4 - (1-\alpha)\frac{S_0}{4}}{\sigma(T_{\mathrm{E}}^4 - T_{\mathrm{A}}^4)} = [8.47, 8.52] \times 10^{-1}$$

$$k = \frac{\epsilon\sigma(2T_{\mathrm{A}}^4 - T_{\mathrm{E}}^4)}{T_{\mathrm{E}} - T_{\mathrm{A}}} = \frac{(2T_{\mathrm{A}}^4 - T_{\mathrm{E}}^4) \times \left[\sigma T_{\mathrm{E}}^4 - (1-\alpha)\frac{S_0}{4}\right]}{(T_{\mathrm{E}}^4 - T_{\mathrm{A}}^4) \times (T_{\mathrm{E}} - T_{\mathrm{A}})} = [3.57, 3.66] \times 10^{-1}\ \mathrm{W/m^2K}$$

B-5 (1.6pt)

$$\epsilon = \frac{\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}}{\sigma (T_E^4 - T_A^4)}$$, Numerical value of $\epsilon = [8.47, 8.52] \times 10^{-1}$

$$k = \frac{(2T_A^4 - T_E^4) \times \left[\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}\right]}{(T_E^4 - T_A^4) \times (T_E - T_A)}$$, Numerical value of $k = [3.57, 3.66] \times 10^{-1}$ $\mathrm{W/m^2K}$

B-6. In order to find the change in the temperatures of the Earth and the atmosphere in terms of ϵ and k, we take the logarithm of both sides of the relations before taking the derivative:

$$ln\,\epsilon = ln\left[\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}\right] - ln\,\sigma - ln\,(T_E^4 - T_A^4)$$

$$ln\,k = ln\,\epsilon + ln\,\sigma + ln(2T_A^4 - T_E^4) - ln(T_E - T_A)$$

$$\frac{1}{\epsilon} = \frac{4\sigma T_E^3 \frac{dT_E}{d\epsilon}}{\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}} - \frac{4T_E^3 \frac{dT_E}{d\epsilon} - 4T_A^3 \frac{dT_A}{d\epsilon}}{T_E^4 - T_A^4}$$

$$0 = \frac{1}{\epsilon} + \frac{8T_A^3 \frac{dT_A}{d\epsilon} - 4T_E^3 \frac{dT_E}{d\epsilon}}{2T_A^4 - T_E^4} - \frac{\frac{dT_E}{d\epsilon} - \frac{dT_A}{d\epsilon}}{T_E - T_A}$$

$$\epsilon\left[\frac{1}{T_E - T_A} + \frac{4T_E^3}{2T_A^4 - T_E^4}\right]\frac{dT_E}{d\epsilon} = 1 + \epsilon\left[\frac{8T_A^3}{2T_A^4 - T_E^4} + \frac{1}{T_E - T_A}\right]\frac{dT_A}{d\epsilon}$$

$$1 + \epsilon\left[\frac{4T_E^3}{T_E^4 - T_A^4} - \frac{4\sigma T_E^3}{\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}}\right]\frac{dT_E}{d\epsilon} = \frac{4T_A^3}{T_E^4 - T_A^4}\epsilon\frac{dT_A}{d\epsilon}$$

Solving this set of linear equations and substituting ϵ in B-5, we find:

$$\frac{dT_{\mathrm{E}}}{d\epsilon}=\frac{\left[\frac{\sigma(T_{\mathrm{E}}^4-T_{\mathrm{A}}^4)}{\sigma T_{\mathrm{E}}^4-(1-\alpha)\frac{S_0}{4}}\right]\left[1+\left(\frac{T_{\mathrm{E}}^4-T_{\mathrm{A}}^4}{4T_{\mathrm{A}}^3}\right)\left[\frac{8T_{\mathrm{A}}^3}{2T_{\mathrm{A}}^4-T_{\mathrm{E}}^4}+\frac{1}{T_{\mathrm{E}}-T_{\mathrm{A}}}\right]\right]}{\left[\frac{1}{T_{\mathrm{E}}-T_{\mathrm{A}}}+\frac{4T_{\mathrm{E}}^3}{2T_{\mathrm{A}}^4-T_{\mathrm{E}}^4}\right]-\left(\frac{\sigma T_{\mathrm{A}}^4-(1-\alpha)\frac{S_0}{4}}{\sigma T_{\mathrm{E}}^4-(1-\alpha)\frac{S_0}{4}}\right)\left(\frac{T_{\mathrm{E}}}{T_{\mathrm{A}}}\right)^3\left[\frac{8T_{\mathrm{A}}^3}{2T_{\mathrm{A}}^4-T_{\mathrm{E}}^4}+\frac{1}{T_{\mathrm{E}}-T_{\mathrm{A}}}\right]}$$

$$\epsilon\frac{dT_{\mathrm{E}}}{d\epsilon}=\frac{1+\left(\frac{T_{\mathrm{E}}^4-T_{\mathrm{A}}^4}{4T_{\mathrm{A}}^3}\right)\left[\frac{8T_{\mathrm{A}}^3}{2T_{\mathrm{A}}^4-T_{\mathrm{E}}^4}+\frac{1}{T_{\mathrm{E}}-T_A}\right]}{\left[\frac{1}{T_{\mathrm{E}}-T_{\mathrm{A}}}+\frac{4T_{\mathrm{E}}^3}{2T_{\mathrm{A}}^4-T_{\mathrm{E}}^4}\right]-\left(\frac{\sigma T_{\mathrm{A}}^4-(1-\alpha)\frac{S_0}{4}}{\sigma T_{\mathrm{E}}^4-(1-\alpha)\frac{S_0}{4}}\right)\left(\frac{T_{\mathrm{E}}}{T_{\mathrm{A}}}\right)^3\left[\frac{8T_{\mathrm{A}}^3}{2T_{\mathrm{A}}^4-T_{\mathrm{E}}^4}+\frac{1}{T_{\mathrm{E}}-T_{\mathrm{A}}}\right]}$$

$$dT_{\mathrm{E}}=\epsilon\frac{dT_{\mathrm{E}}}{d\epsilon}\frac{d\epsilon}{\epsilon}=[5.21\,,5.28]\times10^{-1}\ \mathrm{K}$$

B-6 (1.0pt)

(a)
$$\begin{cases}\epsilon\left[\frac{1}{T_{\mathrm{E}}-T_{\mathrm{A}}}+\frac{4T_{\mathrm{E}}^3}{2T_{\mathrm{A}}^4-T_{\mathrm{E}}^4}\right]\frac{dT_{\mathrm{E}}}{d\epsilon}=1+\epsilon\left[\frac{8T_{\mathrm{A}}^3}{2T_{\mathrm{A}}^4-T_{\mathrm{E}}^4}+\frac{1}{T_{\mathrm{E}}-T_{\mathrm{A}}}\right]\frac{dT_{\mathrm{A}}}{d\epsilon}\\ 1+\epsilon\left[\frac{4T_{\mathrm{E}}^3}{T_{\mathrm{E}}^4-T_{\mathrm{A}}^4}-\frac{4\sigma T_{\mathrm{E}}^3}{\sigma T_{\mathrm{E}}^4-(1-\alpha)\frac{S_0}{4}}\right]\frac{dT_{\mathrm{E}}}{d\epsilon}=\frac{4T_{\mathrm{A}}^3}{T_{\mathrm{E}}^4-T_{\mathrm{A}}^4}\epsilon\frac{dT_{\mathrm{A}}}{d\epsilon}\end{cases}$$

(b) $\delta T_{\mathrm{E}}=[5.21\,,5.28]\times10^{-1}\ \mathrm{K}$

Marking Scheme Q1 (10 points)

Part A (3.0 pt) If the final answer is written then the complete point will be achieved

A-1	$S_0 = \sigma T_S^4 . \left(\frac{R_S}{d}\right)^2$ (0.4pt), [Realizing energy conservation (0.1pt)] Numerical value of $S_0 = 1.35 \times 10^3 \ \mathrm{W/m^2}$ (0.2pt) [more than 4 significant figures (0.1pt)]	0.6 pt
A-2	$T_E = \left(\frac{S_0}{4\sigma}\right)^{\frac{1}{4}} = \sqrt{\frac{R_S}{2d}} T_S$ (0.4pt), [realizing energy balance (0.1pt)] Numerical value of $T_E = 278 \ \mathrm{K}$ (0.2pt) [more than 4 significant figures (0.1pt)]	0.6 pt
A-3	$f(x) = 5(1 - e^{-x}) - x$	0.4 pt
A-4	$x_m = \{4.96, 4.97\}$ (0.3 pt), [more than 4 significant figures (0.2pt)] Numerical value of $b = [2.89, 2.90] \times 10^6 \ \mathrm{nm.K}$ (0.1 pt) [more than 4 significant figures (0.1pt)]	0.4 pt
A-5	$\lambda_{max}^{Sun} = [5.01,\ 5.02] \times 10^2 \ \mathrm{nm}$(0.1 pt), $\lambda_{max}^{Earth} = 1.04 \times 10^4 \ \mathrm{nm}$(0.1 pt) [more than 4 significant figures (0.1pt)]	0.2 pt
A-6	$\gamma = \left(\frac{d}{R_S}\right)^2 \times \left(\frac{T_E}{T_S}\right)^5 = \left(\frac{\lambda_S}{\lambda_E}\right)^5 \times \left(\frac{d}{R_S}\right)^2$ (0.6 pt), [realizing $\tilde{u}_S = \left(\frac{R_S}{d}\right)^2 u_S(\lambda)$(0.3pt)] Numerical value of $\gamma = [1.20, 1.21] \times 10^{-2}$ (0.2 pt) [more than 4 significant figures (0.1pt)]	0.8 pt

Part B (7.0 pt)

B-1	$T_A = \left(\frac{(1-r_A)\frac{S_0}{4}}{\sigma}\right)^{\frac{1}{4}}$ $T_E = \left(\frac{(1-r_A)\frac{S_0}{2}}{\sigma}\right)^{\frac{1}{4}}$ Two correct expressions (0.8 pt) [One correct expression (0.6 pt)] [no correct expression: for each energy balance relation (0.2pt)] Numerical value of $T_A = 2.58 \times 10^2 \ \mathrm{K}$ (0.1 pt) Numerical value of $T_E = 3.07 \times 10^2 \ \mathrm{K}$ (0.1 pt) [more than 4 significant figures (0.1pt)]	1.0 pt
B-2	$\alpha = r_A + \frac{(1-r_A)^2 r_E}{1-r_A r_E}$ (1.4pt) [$\tilde{S}_0 = r_A S_0$ (0.1 pt) $\tilde{S}_1 = (1-r_A)^2 r_E S_0 = \frac{(1-r_A)^2}{r_A} r_E \tilde{S}_0$ (0.3 pt)	1.6 pt

	$\tilde{S}_n = \frac{\tilde{S}_{n-1}}{1-r_A} r_A r_E \times (1-r_A) = r_A r_E \tilde{S}_{n-1} = (r_A r_E)^{n-1}\tilde{S}_1$ (0.5 pt) $\tilde{S} = \sum_{n=0}^{\infty}\tilde{S}_n = \tilde{S}_0 + \tilde{S}_1 \sum_{n=1}^{\infty}(r_A r_E)^{n-1}$ (0.3 pt)] Numerical value of $\alpha = 3.13 \times 10^{-1}$ (0.2pt) [more than 4 significant figures (0.1pt)]	
B-3	$T_E = \left[\frac{(1-\alpha)}{2\sigma(2-\epsilon)} S_0\right]^{\frac{1}{4}}$ (0.6pt) Numerical value of $\epsilon = [8.07, 8.11] \times 10^{-1}$ (0.4pt) [wrong numerical value: correct expression for ϵ (0.2pt)] [more than 4 significant figures (0.3pt)]	1.0 pt
B-4	$\frac{dT_E}{d\epsilon} = \frac{1}{4}\left[\frac{(1-\alpha)S_0}{2\sigma(2-\epsilon)}\right]^{\frac{1}{4}} \frac{1}{(2-\epsilon)}$ (0.6 pt), Numerical value of $\delta T_E = [4.87, 4.92] \times 10^{-1}$ K (0.2pt) [more than 4 significant figures (0.1pt)]	0.8 pt
B-5	$\epsilon = \frac{\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}}{\sigma(T_E^4 - T_A^4)}$ (0.6pt) $k = \frac{(2T_A^4 - T_E^4)\times\left[\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}\right]}{(T_E^4 - T_A^4)\times(T_E - T_A)}$ (0.6pt) [Correct relations for balance of energy (0.3+0.3 pt)] Numerical value of $\epsilon = [8.47, 8.52] \times 10^{-1}$ (0.2pt) Numerical value of $k = [3.57, 3.66] \times 10^{-1}$ W/m^{2}K (0.2pt) [more than 4 significant figures for each one (0.1pt)]	1.6 pt
B-6	(a) (0.4+0.4) $\begin{cases} \epsilon\left[\frac{1}{T_E - T_A} + \frac{4T_E^3}{2T_A^4 - T_E^4}\right]\frac{dT_E}{d\epsilon} = 1 + \epsilon\left[\frac{8T_A^3}{2T_A^4 - T_E^4} + \frac{1}{T_E - T_A}\right]\frac{dT_A}{d\epsilon} \\ 1 + \epsilon\left[\frac{4T_E^3}{T_E^4 - T_A^4} - \frac{4\sigma T_E^3}{\sigma T_E^4 - (1-\alpha)\frac{S_0}{4}}\right]\frac{dT_E}{d\epsilon} = \frac{4T_A^3}{T_E^4 - T_A^4}\epsilon\frac{dT_A}{d\epsilon} \end{cases}$ (0.6 pt) (b) $\delta T_E = [5.21, 5.28] \times 10^{-1}$K (0.2pt) [more than 4 significant figures for each one (0.1pt)]	1.0 pt

A: Paul Trap

A-1. Due to the symmetry, on the z-axis the only non-zero component of electric field is in the z-direction. So:

$$\vec{E}(0,0,z) = E_z(0,0,z)\,\hat{z} = \hat{z}\int \frac{dq}{4\pi\epsilon_0}\frac{1}{(R^2+z^2)} \times \frac{z}{(R^2+z^2)^{\frac{1}{2}}}$$

The element dq is equal to $\lambda R d\phi$ where ϕ is the angle with the x-axis. Thus:

$$E(0,0,z) = \hat{z}\int \frac{\lambda R d\phi}{4\pi\epsilon_0}\frac{z}{(z^2+R^2)^{\frac{3}{2}}} = \hat{z}\frac{\lambda R}{2\epsilon_0}\frac{z}{(z^2+R^2)^{\frac{3}{2}}}$$

For $z \ll R$ this can be written as:

$$E_z(0,0,z) = \frac{\lambda R}{2\epsilon_0}\frac{z}{R^3} = \frac{\lambda z}{2\epsilon_0 R^2}$$

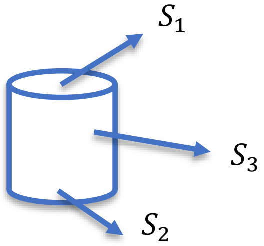
Very close to the z-axis, we can write:

$$E_z(x,y,z) = E_z(0,0,z) + x\frac{\partial E_z}{\partial x}|_{(0,0,z)} + y\frac{\partial E_z}{\partial y}|_{(0,0,z)} + O(x^2,y^2,z^2)$$

Since, there is no difference between x and $-x$ or y and $-y$, it turns out that $\frac{\partial E_z}{\partial x} = \frac{\partial E_z}{\partial y} = 0$. Thus, to the first order in x, y, and z we have:

$$E_z(x,y,z) = \frac{\lambda z}{2\epsilon_0 R^2}$$

Consider a Gaussian surface in the shape of a symmetric cylinder around the z-axis whose bases are parallel with the xy-plane. The cylinder's radius is ρ and its height is $2z$ both of which are small quantities. By Gauss's law we have:



$$0 = \frac{Q_{in}}{\epsilon_0} = \oint \vec{E}.d\vec{S} = \int_{S_1} \vec{E}.d\vec{S} + \int_{S_2} \vec{E}.d\vec{S} + \int_{S_3} \vec{E}.d\vec{S}$$

Integration over S_1 and S_2 gives:

$$\int_{S_1} \vec{E}.d\vec{S} = \int_{S_2} \vec{E}.d\vec{S} = \pi\rho^2 \times \frac{\lambda z}{2\epsilon_0 R^2}.$$

Integration over S_3 involves the ρ-component for which we can write the following expansion:

$$E_\rho(z,\rho,\phi) = E_\rho(0,\rho,\phi) + z\frac{\partial E_\rho}{\partial z}|_{(0,\rho,\phi)} + O(z^2)$$

We have $0 = \frac{\partial E_\rho}{\partial z}|_{(0,\rho,\phi)}$ due to symmetry between z and $-z$, hence, $E_\rho(z,\rho,\phi) = E_\rho(0,\rho,\phi)$ up to the first order. Axial symmetry also implies $\frac{dE_\rho}{d\phi} = 0$. Consequently:

$$\int_{S_3} \vec{E}.d\vec{S} = E_\rho(0,\rho,0) \times 2z \times 2\pi\rho$$

So, Gauss's law implies:

$$0 = E_\rho \times 4\pi z\rho + 2\pi\rho^2 \frac{\lambda z}{2\epsilon_0 R^2}$$

Therefore, E_ρ will be:

$$E_\rho = -\frac{\lambda\rho}{4\epsilon_0 R^2}$$

In the cylindrical coordinate we will have:

$$\vec{E}(\rho,\phi,z) = -\frac{\lambda\rho}{4\epsilon_0 R^2}\hat{\rho} + \frac{\lambda z}{2\epsilon_0 R^2}\hat{z}$$

In cartesian coordinates we will have:

$$\vec{E}(x,y,z) = \frac{\lambda}{4\epsilon_0 R^2}(-x,-y,2z)$$

Since the ring is positively charged, the equilibrium in the x and y directions are stable, while the equilibrium in the z-direction is unstable. The equations of motion in the x and y directions read:

$$m\ddot{x} = qE_x = -\frac{q\lambda}{4\epsilon_0 R^2}x$$

$$m\ddot{y} = qE_y = -\frac{q\lambda}{4\epsilon_0 R^2}y$$

Therefore, the frequencies of small oscillations are:

$$\omega_x^2 = \omega_y^2 = \frac{q\lambda}{4\epsilon_0 R^2 m}$$

A-1 (1.5 pt)

(a) $\vec{E}(x,y,z) = \frac{-\lambda x}{4\epsilon_0 R^2}\hat{x} + \frac{-\lambda y}{4\epsilon_0 R^2}\hat{y} + \frac{\lambda z}{2\epsilon_0 R^2}\hat{z}$

(b) $\omega_x = \omega_y = \sqrt{\frac{Q\lambda}{4\epsilon_0 R^2 m}}$

A-2.

The force in the z-direction is:

$$F_z = qE_z = \frac{Q\lambda z}{2\epsilon_0 R^2} = \frac{Q}{2\epsilon_0 R^2}\ \lambda_0 z + \frac{Qu}{2\epsilon_0 R^2}\cos\Omega t\, z$$

the equation of motion can thus be written as:

$$\ddot{z} = \left(\frac{Q\lambda_0}{2\epsilon_0 R^2 m} + \frac{Qu}{2\epsilon_0 R^2 m}\cos\Omega t\right) z$$

Therefore:

$$k = \sqrt{\frac{Q\lambda_0}{2\epsilon_0 R^2 m}} \quad , \quad a = \frac{Qu}{2\epsilon_0 R^2 m\Omega^2}$$

A-2 (0.4 pt)

$k = \sqrt{\frac{Q\lambda_0}{2\epsilon_0 R^2 m}}$, $a = \frac{Qu}{2\epsilon_0 R^2 m\Omega^2}$

A.3.

$$z = p(t) + q(t) \quad \rightarrow \quad \ddot{p} + \ddot{q} = (k^2 + a\Omega^2\cos\Omega t)(p + q)$$

1. We are assuming that p is almost constant, $\ddot{p} \simeq 0$.
2. According to the assumptions $k^2 \ll a\Omega^2$ and $q \ll p$ we can ignore k^2 in the first term on the right-hand side of the equation and q in the second term.

hence, the equation of motion can be simplified as follows:

$$\ddot{q} = pa\Omega^2\cos\Omega t.$$

As we have assumed that p is a constant, the second derivative of q is just proportional to $\cos\Omega t$ which gives:

$$q = -pa\cos\Omega t + c_1 t + c_2.$$

Since q is supposed to remain small, c_1 must vanish. Also $c_2 = 0$ because the mean value of q is supposed to remain zero. Therefore:

$$q = -pa\cos\Omega t$$

A-3 (1.8 pt)
(a) $\ddot{q}(t) = pa\Omega^2\cos\Omega t$
(b) $q(t) = -pa\cos\Omega t$

A-4. Using the final result for q the equation of motion for p reads:

$$\ddot{p} + pa\Omega^2\cos\Omega t = (k^2 + a\Omega^2\cos\Omega t)(p - ap\cos\Omega t)$$

Which gives:

$$\ddot{p} = k^2 p - ak^2 p\cos\Omega t - a^2\Omega^2 p\cos^2\Omega t$$

Averaging over one period, we'll have:

$$\langle\cos\Omega t\rangle = 0 \qquad , \qquad \langle\cos^2\Omega t\rangle = \frac{1}{2}$$

and:

$$\ddot{p} = \left(k^2 - \frac{a^2\Omega^2}{2}\right)p.$$

In order for the motion to be stable, the expression inside the parentheses should be negative, i.e.

$$\frac{a^2\Omega^2}{2} > k^2 \qquad \rightarrow \qquad \Omega > \sqrt{2}\frac{k}{a}$$

A-4 (1.5 pt)
(a) $\ddot{p}(t) = \left(k^2 - \frac{a^2\Omega^2}{2}\right)p$
(b) $\Omega > \sqrt{2}\frac{k}{a}$

A.5. With the given data we have:

$$k = \sqrt{\frac{Q\lambda_0}{2\epsilon_0 R^2 m}} = 2 \times 10^5 \text{ rad/s}$$

$$a = 0.04 \quad \rightarrow \quad \Omega_{\text{min}} = 7 \times 10^6 \text{ rad/s}$$

which is in the range of radio waves.

A-5 (0.4 pt)
$k = 2 \times 10^5$ rad/s
$\Omega_{\text{min}} = 7 \times 10^6$rad/s

B: Doppler Cooling

B-1. From the uncertainty principle we know:

$$\Delta E \times \Delta t \simeq \hbar$$

Here Δt is the time τ and $\Delta E = \hbar\Delta\omega$. So:

$$\hbar\Delta\omega \times \tau \simeq \hbar \quad \rightarrow \quad \Delta\omega \simeq \frac{1}{\tau} = \Gamma$$

B-1 (0.5 pt)
$\Gamma = \frac{1}{\tau}$

B-2. We denote the forward and backward collision rates by s_+ and s_- respectively. Let us proceed in the atom's frame of reference. Ignoring the terms of the order $\frac{v^2}{c^2}$, the Doppler effect can be written in the following form:

$$\omega' = \omega\left(1 + \frac{v}{c}\right)$$

Taking the atom's velocity in the positive x-direction, we have:

$$\omega_+ = \omega_L\left(1+\frac{v}{c}\right)$$

$$\omega_- = \omega_L\left(1-\frac{v}{c}\right)$$

So:

$$s_+ = s_L + \alpha\left(\omega_L\left(1+\frac{v}{c}\right) - \omega_L\right) = s_L + \alpha\omega_L\frac{v}{c}$$

$$s_- = s_L + \alpha\left(\omega_L\left(1-\frac{v}{c}\right) - \omega_L\right) = s_L - \alpha\omega_L\frac{v}{c}$$

The momentum transfer per unit time from the oncoming photons to the atom is equal to:

$$\pi_+ = s_+ \times (-\hbar k_+)$$

For the backward photons we have:

$$\pi_- = s_- \times (+\hbar k_-)$$

Where $k_\pm = \frac{\hbar\omega_\pm}{c}$.

The total momentum transferred to the atom per unit time is equal to:

$$\pi_+ + \pi_- = -2\hbar k_L \frac{v}{c}\,\omega_L\alpha\left(1+\frac{s_L}{\alpha\omega_L}\right)$$

Where with the approximation $s_L \ll \alpha\omega_L$, we will arrive at:

$$\pi_+ + \pi_- = -2\hbar k_L \frac{v}{c}\,\omega_L\alpha$$

Note that as the atom is heavy, its velocity almost doesn't change after the absorption of the photon. Therefore, there will be almost no Doppler shifting in the re-emitted photon and hence, on average there will be no momentum transfer to the atom during the re-emission process.

The above expression is, in fact, the force. Since $v > 0$, we have:

$$F = -(2\alpha\hbar k_L^2)v$$

The same result holds for $v < 0$. This is in the atom's reference frame. However, as we have kept only up to the first order in v/c, the same result holds in the lab frame:

$$F = -(2\alpha\hbar k_L^2)v$$

B-2 (1.7 pt)

$s_+ = s_L + \alpha\omega_\mathrm{L}\frac{v}{c}$

$s_- = s_L - \alpha\omega_\mathrm{L}\frac{v}{c}$

$\pi_+ = s_+ \times (-\hbar k_+)$

$\pi_- = s_- \times (+\hbar k_-)$

$F = -(2\alpha\hbar k_\mathrm{L}^2)v$

B-3. The atom's momentum before the collision is zero. After the collision it will be (assuming the photon's momentum is in the x-direction):

$$P_1 = \hbar k_\mathrm{L}$$

After re-emitting the photon, we may have two equally likely outcomes for the final momentum:

1. The photon is emitted in the positive x-direction which causes the atom's momentum to become zero
2. The photon is emitted in the negative x-direction which causes the atom's momentum to become: $P_\mathrm{f} = +2\hbar k_\mathrm{L}$

Thus, the mean final energy is equal to:

$$\langle E_\mathrm{f}\rangle = \langle\frac{P_\mathrm{f}^2}{2m}\rangle = \frac{1}{2}\times 0 + \frac{1}{2}\times\frac{4\hbar^2 k_\mathrm{L}^2}{2m} = \frac{\hbar^2 k_\mathrm{L}^2}{m}$$

This process occurs during the time τ. So, the input power (the power gained by the atom as a result of this process) is equal to:

$$P_\mathrm{in} = \frac{\hbar^2 k_\mathrm{L}^2}{m\tau}$$

B-3 (1.0 pt)

$P_\mathrm{in} = \frac{\hbar^2 k_\mathrm{L}^2}{m\tau}$

B.4. The output power (the power lost by the atom through collision with laser photons) can be written as:

$$P_{\mathrm{out}} = F \cdot v = -2\alpha\hbar k_{\mathrm{L}}^2 v^2$$

At equilibrium we should have:

$$P_{\mathrm{out}} + P_{\mathrm{in}} = 0 \quad \rightarrow \quad \frac{\hbar^2 k_{\mathrm{L}}^2}{m\tau} = 2\alpha\hbar k_{\mathrm{L}}^2 \overline{v^2} \quad \rightarrow \quad \overline{v^2} = \frac{\hbar\Gamma}{2\alpha m}$$

And the temperature of this system is equal to:

$$\frac{1}{2} m\overline{v^2} = \frac{1}{2} k_{\mathrm{B}} T \quad \rightarrow \quad T = \frac{\hbar\Gamma}{2\alpha k_{\mathrm{B}}}$$

B-4 (0.8 pt)

$P_{\mathrm{out}} = -2\alpha\hbar k_{\mathrm{L}}^2 v^2$

$\overline{v^2} = \frac{\hbar\Gamma}{2\alpha m}$

$T = \frac{\hbar\Gamma}{2\alpha k_{\mathrm{B}}}$

B-5. Considering the given data:

$$T = \frac{1\,055 \times 10^{\ 34}\ \mathrm{J.s}}{2 \times 4 \times 1\,381 \times 10^{\ 23}\ \mathrm{J/K} \times 5 \times 10^{\ 9}\ \mathrm{s}} = 2 \times 10^{-4}\ \mathrm{K}$$

B-5 (0.4 pt)

$T = 2 \times 10^{-4}\ \mathrm{K}$

Marking Scheme Q2 (10 points)

Part A (5.6 pt)

A-1		1.5 pt
(a)	$\vec{E}(x,y,z) = \frac{-\lambda x}{4\epsilon_0 R^2}\hat{x} + \frac{-\lambda y}{4\epsilon_0 R^2}\hat{y} + \frac{\lambda z}{2\epsilon_0 R^2}\hat{z}$ (1.0 pt) [z-component (0.5 pt), x- and y- components (0.5 pt), wrong coefficient for each component (-0.1 pt), wrong sign for each component (-0.2 pt)]	
(b)	$\omega_x = \omega_y = \sqrt{\frac{Q\lambda}{4\epsilon_0 R^2 m}}$ (0.5 pt)	
A-2	$a = \frac{Qu}{2\epsilon_0 R^2 m \Omega^2}$ (0.2 pt) $k = \sqrt{\frac{Q\lambda_0}{2\epsilon_0 R^2 m}}$ (0.2 pt)	0.4 pt
A-3	$\ddot{q} = pa\Omega^2 \cos \Omega t$ (1.0 pt) [each of the 3 approximations (0.3 pt), the final equation (0.1pt)] $q = -pa \cos \Omega t$ (0.8 pt) [general solution (0.4 pt), fixing the free parameters in the general solution each (0.2 pt)]	1.8 pt
A-4	$\ddot{p}(t) = \left(k^2 - \frac{a^2\Omega^2}{2}\right)p$ (1.2 pt) [Correct approach (0.6 pt), Correct result (0.6 pt)] $\Omega > \sqrt{2}\frac{k}{a}$ (0.3 pt)	1.5 pt
A-5	$k = 2 \times 10^5$ rad/s (0.2 pt) $\Omega_{\text{min}} \simeq 7 \times 10^6$ rad/s (0.2 pt) [inappropriate number of significant figures (-0.1 pt)]	0.4 pt

Part B (4.4 pt)

B-1	$\Gamma = \frac{1}{\tau}$ (0.5 pt) [Answers with different numerical coefficients should be considered as correct answers]	0.5 pt

B-2	$s_+ = s_L + \alpha\omega_L \frac{v}{c}$ (0.5 pt) $s_- = s_L - \alpha\omega_L \frac{v}{c}$ (0.5 pt) [correct Doppler shift each (0.3 pt), final answer each (0.2 pt)] $\pi_+ = s_+ \times (-\hbar k_+)$ (0.1 pt) $\pi_- = s_- \times (+\hbar k_-)$ (0.1 pt) $F = -(2\alpha\hbar k_L^2)v$ (0.5 pt)	1.7 pt
B-3	$\begin{cases} p = 0 \\ p = +2\hbar k_L \end{cases}$ (0.5 pt) [one correct answer (0.3 pt)] $P_{in} = \frac{\hbar^2 k_L^2}{m\tau}$ (0.5 pt)	1.0 pt
B-4	$P_{out} = -2\alpha\hbar k_L^2 v^2$ (0.3 pt) $\overline{v^2} = \frac{\hbar\Gamma}{2\alpha m}$ (0.3 pt) $T = \frac{\hbar\Gamma}{2\alpha k_B}$ (0.2 pt) [Answers with different numerical coefficients should be considered as correct answers]	0.8 pt
B-5	$T = 2 \times 10^{-4}$ K (0.4 pt) [according to the coefficient used in the part B.4, the resulting temperature might be different.]	0.4 pt

A. A Binary System

A-1. Assume $a_1$and a_2, are respectively, the distances of $M_1$and M_2 from the center of mass:

$$\begin{cases} M_1 a_1 = M_2 a_2 \\ a_1 + a_2 = a \end{cases} \rightarrow a_1 = \frac{M_2}{M} a \ , a_2 = \frac{M_1}{M} a \ : M = M_1 + M_2$$

In the rotating coordinate system, a centrifugal potential has to be added to the gravitational potential of the two masses:

$$U = -\frac{1}{2}\omega^2 r^2 \qquad \omega = \sqrt{\frac{GM}{a^3}}$$

$$\varphi(x,y) = -\frac{GM_1}{\sqrt{(x+a_1)^2 + y^2}} - \frac{GM_2}{\sqrt{(x-a_2)^2 + y^2}} - \frac{1}{2}\,\omega^2(x^2 + y^2)$$

$$\varphi(x,y) = -\frac{GM_1}{\sqrt{\left(x + \frac{M_2}{M} a\right)^2 + y^2}} - \frac{GM_2}{\sqrt{\left(x - \frac{M_1}{M} a\right)^2 + y^2}} - \frac{1}{2}\frac{GM}{a^3}(x^2 + y^2)$$

A-1 (1.0 pt)

$$\varphi(x,y) = -\frac{GM_1}{\sqrt{\left(x + \frac{M_2}{(M_1+M_2)} a\right)^2 + y^2}} - \frac{GM_2}{\sqrt{\left(x - \frac{M_1}{(M_1+M_2)} a\right)^2 + y^2}} - \frac{1}{2}\frac{G(M_1+M_2)}{a^3}(x^2 + y^2)$$

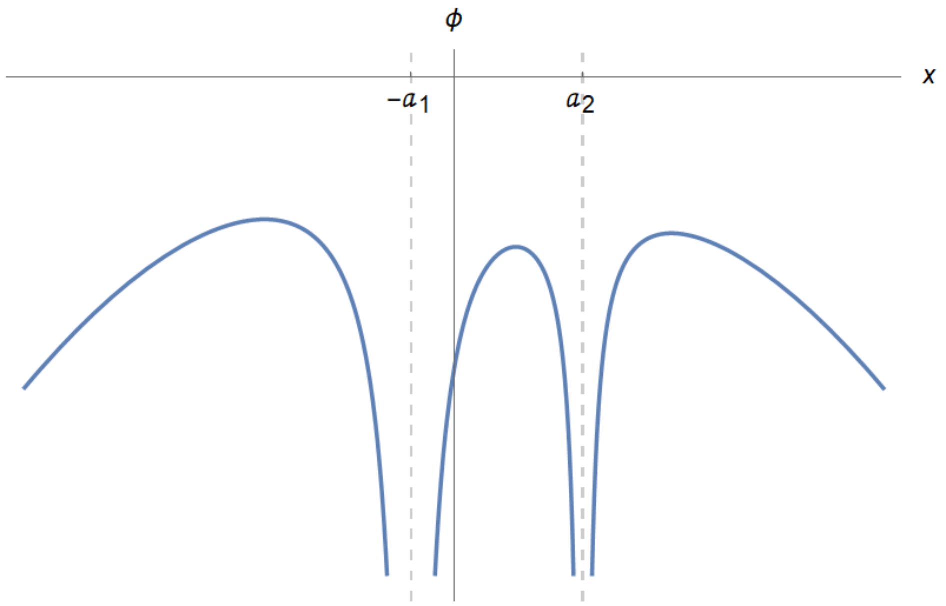
A-2. We set $y = 0$ in the previous equation, and obtain:

$$\varphi(x,0) = -\frac{GM_1}{\left|x + \frac{M_2}{M} a\right|} - \frac{GM_2}{\left|x - \frac{M_1}{M} a\right|} - \frac{1}{2}\frac{GM}{a^3}x^2$$

We draw the diagram noting that:

1. The function has asymptotes at $x = -a_1$ and $x = a_2$, and it tends to $-\infty$ at both sides of these asymptotes.
2. The function has three maxima which are called Lagrange points.
3. The function goes to $-\infty$ for $x \rightarrow \pm\infty$

A-2 (0.7 pt)



A-3. Let $\bar{x} = x/a$, and denote the Lagrange point in the middle (between $\bar{x} = 0$ and $\bar{x} = 0.75$) by $\bar{x}_0$, we have $\frac{d\varphi}{d\bar{x}}(\bar{x}_0) = 0$. Using the given ratios:

$$\varphi(\bar{x}, 0) = \frac{GM}{a}\left[-\frac{\frac{3}{4}}{\left(\bar{x} + \frac{1}{4}\right)} + \frac{\frac{1}{4}}{(\bar{x} - \frac{3}{4})} - \frac{1}{2}\bar{x}^2\right]$$

Let $f(\bar{x}) = \frac{a}{GM}\frac{d\varphi}{d\bar{x}}$, then we have to solve for $f(\bar{x}_0) = 0$. We have $f(0) > 0$ and $f(0.5) < 0$, so the answer lies between 0 and 0.5. For the midpoint, we have $f(\bar{x}_0 = 0.25) > 0$ so $0.25 < \bar{x}_0 < 0.5$, so by trial and error:

$$\begin{cases} f(0) > 0 \\ f(0.5) < 0 \end{cases} \to f(0.25) > 0 \to 0.25 < \bar{x}_0 < 0.5 \to f(0.375) < 0 \to \cdots \to 0.358 < \bar{x}_0 < 0.361$$

$$\to f(0.360) > 0 \to 0.360 < \bar{x}_0 < 0.361 \to \frac{x_0}{a} = \bar{x}_0 \approx 0.36$$

So, up to two significant figures the answer is 0.36.

A-3 (0.5 pt)

$\frac{x_0}{a} = 0.36$

A-4. The angular momentum of the system is:

$$J = \mu a V = \mu a^2 \omega = \frac{M_1 M_2}{M} a^2 \sqrt{\frac{GM}{a^3}} = \sqrt{\frac{G M_1^2 M_2^2}{M}}\, a,$$

where μ is the reduced mass and V is the relative velocity of the two point masses. Taking the logarithm of both sides we'll have:

$$\ln J = \frac{1}{2}\left[\ln\frac{G}{M} + 2\ln M_1 + 2\ln M_2 + \ln a\right]$$

For slowly-varying quantities we'll obtain:

$$\frac{\dot{J}}{J} = \frac{\dot{M}_1}{M_1} + \frac{\dot{M}_2}{M_2} + \frac{1}{2}\frac{\dot{a}}{a}$$

because the total mass is a constant and $\dot{M}_1 + \dot{M}_2 = 0$; therefore:

$$\frac{\dot{a}}{a} = -2\frac{\dot{M}_1}{M_1}\left(1 - \frac{M_1}{M_2}\right) \quad \rightarrow \quad \dot{a} = -2\beta a\left(\frac{1}{M_1} - \frac{1}{M_2}\right)$$

For the period we'll have:

$$P = 2\pi\sqrt{\frac{a^3}{GM}} \;\rightarrow\; \frac{\dot{P}}{P} = \frac{3}{2}\frac{\dot{a}}{a} = -3\frac{\dot{M}_1}{M_1}\left(1 - \frac{M_1}{M_2}\right) \;\rightarrow\; \dot{P} = -6\pi\sqrt{\frac{a^3}{GM}}\beta\left(\frac{1}{M_1} - \frac{1}{M_2}\right)$$

A-4 (0.6 pt)
$\dot{a} = -2\beta a\left(\frac{1}{M_1} - \frac{1}{M_2}\right)$ $\dot{P} = -6\pi\sqrt{\frac{a^3}{GM}}\beta\left(\frac{1}{M_1} - \frac{1}{M_2}\right)$

A-5. In an infinitesimally thin ring with an inner radius of r and an outer radius $r + dr$, energy is leaving at a rate of $-\frac{GM_1\beta}{2r}$ and entering at a rate $-\frac{GM_1\beta}{2r} + \frac{GM_1\beta}{2r^2}dr$. For the ring to stay in equilibrium, the excess energy of $\frac{GM_1\beta}{2r}dr$ per unit time must leave the system as radiation, so:

$$dP = \frac{GM_1\beta}{2r^2}dr = \sigma T^4 2(2\pi r dr) = 4\pi\sigma T^4 dr \;\rightarrow\; T = \left(\frac{GM_1\beta}{8\pi\sigma r^3}\right)^{\frac{1}{4}}$$

A-5 (1.0 pt)
$T = \left(\frac{GM_1\beta}{8\pi\sigma r^3}\right)^{\frac{1}{4}}$

A-6. From $P = 2\pi\sqrt{\frac{a^3}{GM}}$ we'll have:

$$a = \left[\frac{P^2 G(M_S + M_{NS})}{4\pi^2}\right]^{\frac{1}{3}}$$

Using the result of Part A.5, the temperature is:

$$T = \left(\frac{GM_{NS}\beta}{8\pi\sigma r^3}\right)^{\frac{1}{4}} = \left(\frac{500\pi\, M_{NS}\,\beta}{\sigma P^2(M_S + M_{NS})}\right)^{\frac{1}{4}} = 9 \times 10^3 K$$

A-6 (0.5 pt)
$T = 9 \times 10^3\ K$

A.7. For the system to remain bounded, the total mechanical energy of the system must be negative:

$$E' = \frac{1}{2}\mu' v'^2 - \frac{GM_1' M_2}{a} < 0 \rightarrow v' < \sqrt{\frac{2G(M_1' + M_2)}{a}}$$

For an isotropic explosion, we would have $v' = v = \sqrt{\frac{GM}{a}}$ therefore:

$$\sqrt{\frac{G(M_1 + M_2)}{a}} < \sqrt{\frac{2G(M_1' + M_2)}{a}}$$

and:

$$\frac{M_1 - M_2}{2} < M_1'$$

A-7 (0.7 pt)
$v'_{\max} = \sqrt{\frac{2G(M_1'+M_2)}{a}}$ $M'_{1\min} = \frac{M_1 - M_2}{2}$

B. Analysis of the stability of a star

B-1. Using Newton's law of gravity:

$$g = -\frac{4\pi G \int_0^r r'^2 \rho dr'}{r^2} \overset{\rho \cong \rho_c}{\cong} -\frac{4\pi G \rho_c r}{3}$$

B-1 (0.2 pt)

$g = -\frac{4\pi G\rho_c r}{3}$

B-2. Balance of forces for a differential element of volume with a surface area of A and thickness Δr between radii r and $r + \Delta r$ is as follows:

$$\vec{F} = -\frac{GM(\vec{r})\rho}{r^2}\mathrm{A}\,\Delta r - \Delta pA = 0$$

in which $M(r)$ is the mass of the part of the star confined within the radius r. As Δr is small, we can write:

$$\frac{G\rho}{r^2}\left(\int 4\pi r'^2\rho(r')dr'\right) = -\frac{dp(r)}{dr} = -K\gamma\rho^{\gamma-1}\frac{d\rho}{dr}$$

Multiplying both sides of the equation by $\frac{r^2}{4\pi G\rho}$ and taking the derivative once again, we get:

$$\frac{d}{dr}\left[r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right] + \frac{4\pi Gr^2}{K\gamma}\rho(r) \; = 0$$

B-2 (0.6 pt)

$h_1(\rho, r) = r^2\rho^{\gamma-2}$

$h_2(r) = \frac{4\pi Gr^2}{K\gamma}$

B-3.

$$[\rho_c] = ML^{-3}, \quad [p_c] = ML^{-1}T^{-2}, \quad [G] = M^{-1}L^3T^{-2}$$

$$[G^l p_c^m \rho_c^n] = (M^{-1}L^3T^{-2})^l(ML^{-1}T^{-2})^m(ML^{-3})^n = L$$

$$\begin{cases} -l + n + m = 0 \\ 3l - 3n - m = 1 \\ -2l - 2m = 0 \end{cases} \rightarrow \begin{cases} l = -\frac{1}{2} \\ m = \frac{1}{2} \\ n = -1 \end{cases} \rightarrow \; r_0 = G^{-\frac{1}{2}}p_c^{\frac{1}{2}}\rho_c^{-1}$$

B-3 (0.4 pt)

$r_0 = G^{-\frac{1}{2}}p_c^{\frac{1}{2}}\rho_c^{-1}$

B-4.

$$\frac{K\gamma\rho_c^{\gamma-1}}{4\pi G r_0^2 x^2}\frac{d}{dx}\left[x^2u^{\gamma-2}\frac{du}{dx}\right] = -\rho_c u(r)$$

$$\frac{K\gamma\rho_c^{\gamma-2}}{4\pi G r_0^2 x^2}\frac{d}{dx}\left[x^2u^{\gamma-2}\frac{du}{dx}\right] = \frac{\gamma}{4\pi x^2}\frac{d}{dx}\left[x^2u^{\gamma-2}\frac{du}{dx}\right] = -u$$

$$\frac{d}{dx}\left[x^2u^{\gamma-2}\frac{du}{dx}\right] + \frac{4\pi x^2}{\gamma}u = 0$$

B-4 (0.3 pt)
$A_1(u,x) = x^2u^{\gamma-2}$
$A_2(x) = \frac{4\pi x^2}{\gamma}$

B-5.

$$\gamma = 2 \;\rightarrow\; \frac{d}{dx}\left[x^2\frac{du}{dx}\right] = -2\pi x^2 u(x) \rightarrow f''(x) = -2\pi f(x) \rightarrow\; f(x) = \frac{\sin\left(\sqrt{2\pi}x\right)}{\sqrt{2\pi}}$$

B-5 (0.6 pt)
$f(x) = \frac{\sin(\sqrt{2\pi}x)}{\sqrt{2\pi}}$

B.6.

$$\frac{d^2u}{dx^2} + \frac{(\gamma-2)}{u}\left(\frac{du}{dx}\right)^2 + \frac{2}{x}\left(\frac{du}{dx}\right) + \frac{4\pi}{\gamma}u^{3-\gamma} = 0$$

$$u'(0) = 0 \;\;,\;\; \lim_{x\to 0}\frac{u'(x)}{x} = u''(0)$$

$$u''(0) + 2u''(0) + \frac{4\pi}{\gamma} = 0 \;\;\rightarrow\;\; \gamma = -\frac{4\pi}{3u''(0)}$$

$$\gamma \sim [1.64, 1.70]$$

B-6 (0.8 pt)
$\gamma = [1.64, 1.70]$

B.7.

$$M(r) = \int_0^{\tilde{r}(r,t)} 4\pi r'^2 \tilde{\rho}(r',t)dr' = \int_0^{r} 4\pi r'^2 \rho(r')dr'$$

$$4\pi r^2 \rho(r) = 4\pi \tilde{r}^2 \tilde{\rho}(\tilde{r},t)\frac{\partial \tilde{r}}{\partial r} \rightarrow \frac{\tilde{\rho}}{\rho} = \frac{r^2}{\tilde{r}^2}\left(\frac{\partial \tilde{r}}{\partial r}\right)^{-1} = (1+\epsilon)^{-3} \cong 1 - 3\epsilon$$

$$\frac{\tilde{g}}{g} = \frac{\frac{GM}{\tilde{r}^2}}{\frac{GM}{r^2}} = \frac{\frac{1}{\tilde{r}^2}}{\frac{1}{r^2}} = (1+\epsilon)^{-2} \cong 1 - 2\epsilon$$

B-7 (0.9 pt)

$\tilde{g} \simeq g(1-2\epsilon)$

$\tilde{\rho} \simeq \rho(1-3\epsilon)$

B-8. we have

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = \tilde{\rho}\left(\tilde{g} - \ddot{\tilde{r}}\right)$$

And

$$\tilde{p} = K\tilde{\rho}^{\gamma}$$

So:

$$\ddot{\tilde{r}} = \tilde{g} - \frac{\left(\frac{\partial \tilde{p}}{\partial \tilde{r}}\right)}{\tilde{\rho}} = \tilde{g} - K\gamma\tilde{\rho}^{\gamma-2}\frac{\partial \tilde{\rho}}{\partial \tilde{r}}$$

B-8 (0.6 pt)

$$\frac{d^2\tilde{r}}{dt^2} = \tilde{g} - K\gamma\tilde{\rho}^{\gamma-2}\frac{\partial \tilde{\rho}}{\partial \tilde{r}}$$

B.9. Using of the results in B.7 and B.8, we have:

$$\frac{d^2\tilde{r}}{dt^2} = \ddot{\tilde{r}} = \tilde{g} - K\gamma\tilde{\rho}^{\gamma-2}\frac{\partial \tilde{\rho}}{\partial \tilde{r}} = g(1-2\epsilon) - K\gamma\rho^{\gamma-2}\frac{\partial \rho}{\partial r}\left(\frac{(1-3\epsilon)^{\gamma-1}}{(1+\epsilon)}\right)$$
$$= g(1-2\epsilon) - K\gamma\rho^{\gamma-2}\frac{\partial \rho}{\partial r}(1 - 3(\gamma-1)\epsilon - \epsilon)$$

Equilibrium requires:

$$g - K\gamma\rho^{\gamma-2}\frac{\partial \rho}{\partial r} = 0 \Rightarrow K\gamma\rho^{\gamma-2}\frac{\partial \rho}{\partial r} = g$$

therefore:

$$\ddot{\tilde{r}} = r\ddot{\epsilon} = g(1-2\epsilon) - g(1-3(\gamma-1)\epsilon - \epsilon) = g(3\gamma-4)\epsilon$$

and:

$$\ddot{\epsilon} = \frac{g}{r}(3\gamma - 4)\epsilon$$

$$\ddot{\epsilon} = -\frac{4\pi G\rho_c}{3}(3\gamma - 4)\epsilon$$

Stability requires that:

$$3\gamma - 4 > 0 \Rightarrow \gamma > \frac{4}{3}$$

and the angular velocity of the oscillations will be:

$$\omega = \sqrt{\frac{4\pi G\rho_c}{3}(3\gamma - 4)}$$

B-9 (0.6 pt)

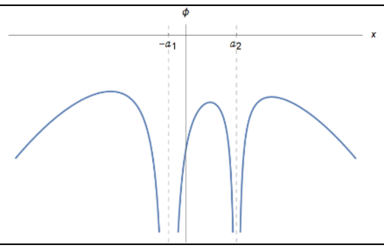
$$\ddot{\epsilon} = -\frac{4\pi G\rho_c}{3}(3\gamma - 4)\epsilon$$

$$\gamma_{\min} = \frac{4}{3}$$

$$\omega = \sqrt{\frac{4\pi G\rho_c}{3}(3\gamma - 4)}$$

Marking Scheme Q3 (10 points)

Part A (5.0 pt)

A-1	$\Phi(x,y) = -\frac{GM_1}{\sqrt{\left(x+\frac{M_2}{(M_1+M_2)}a\right)^2+y^2}} - \frac{GM_2}{\sqrt{\left(x-\frac{M_1}{(M_1+M_2)}a\right)^2+y^2}} - \frac{1}{2}\frac{G(M_1+M_2)}{a^3}(x^2+y^2)$ [Gravitational part (0.5 pt)] [Centrifugal part (0.5 pt)]	1.0 pt
A-2	[Correct behavior at infinity (0.1 pt)] [Three maximums (0.3 pt)] [Two vertical asymptotes (0.3 pt)]  	0.7 pt
A-3	$\frac{x_0}{a} = 0.36$ [In case of obtaining correct equation but not solving it (0.2 pt)] [Obtaining the numerical result with one decimal figure (0.3 pt)]	0.5 pt
A-4	$\dot{a} = -2\beta a\left(\frac{1}{M_1}-\frac{1}{M_2}\right)$ (0.3 pt) $\dot{P} = -6\pi\sqrt{\frac{a^3}{GM}}\beta\left(\frac{1}{M_1}-\frac{1}{M_2}\right)$ (0.3 pt) [Only correct approach (conservation of momentum) (0.2 pt)]	0.6 pt
A-5	$T = \left(\frac{GM_1\beta}{8\pi\sigma r^3}\right)^{\frac{1}{4}}$ [Correct approach (Energy relation) (0.5 pt)] [Correct solution (0.5 pt)]	1.0 pt
A-6	$a = \left[\frac{P^2G(M_S+M_{NS})}{4\pi^2}\right]^{\frac{1}{3}}$ (0.3 pt) $T = \left(\frac{500\pi M_{NS}\beta}{\sigma P^2(M_S+M_{NS})}\right)^{\frac{1}{4}}$ (0.1 pt) $T = 9\times10^3\ \mathrm{K}$ (0.1 pt) [If the final answer for T is correct the complete pt will be given]	0.5 pt
A-7	$E' = \frac{1}{2}\mu' v'^2 - \frac{GM_1'M_2}{a} < 0$ (0.2 pt) $v'_{max} = \sqrt{\frac{2G(M_1'+M_2)}{a}}$ (0.2 pt) $v' = v$ (0.2 pt) $M'_{1min} = \frac{M_1-M_2}{2}$ (0.1 pt)	0.7 pt

Part B (5.0 pt)

B-1	$g=-\frac{4\pi G\rho_c r}{3}$	0.2 pt
B-2	$h_1(\rho,r)=r^2\rho^{\gamma-2}$ $h_2(r)=\frac{4\pi G r^2}{K\gamma}$ $[\vec{F}=-\frac{GM(\vec{r})\rho}{r^2}A\,\Delta r-\Delta pA=0$ (0.3 pt)]	0.6 pt
B-3	$r_0=G^{-\frac{1}{2}}p_c^{\frac{1}{2}}\rho_c^{-1}$	0.4 pt
B-4	$A_1(u,x)=x^2u^{\gamma-2}$ $A_2(x)=\frac{4\pi x^2}{\gamma}$ The answer would be correct up to a constant coefficient	0.3 pt
B-5	$f(x)=A\sin(\sqrt{2\pi}x)+B\cos(\sqrt{2\pi}x)$ (0.3 pt) $A=\frac{1}{\sqrt{2\pi}}$ (0.2 pt) & $B=0$ (0.1 pt)	0.6 pt
B-6	$u'(0)=0$ (0.1 pt) $\lim_{x\to 0}\frac{u'(x)}{x}=u''(0)$ (0.4 pt) $\gamma=-\frac{4\pi}{3u''(0)}$ (0.2 pt) $\gamma\sim 1.66$ (0.1 pt)	0.8 pt
B-7	$\tilde{\rho}\simeq\rho(1-3\epsilon)$ (0.6 pt) $[\tilde{\rho}=\rho(1+\epsilon)^{-3}$(0.4pt)] $\tilde{g}\simeq g(1-2\epsilon)$ (0.3 pt) $[\tilde{g}=g(1+\epsilon)^{-2}$(0.2pt)]	0.9 pt
B-8	$\ddot{\tilde{r}}=\tilde{g}-k\gamma\tilde{\rho}^{\gamma-2}\frac{\partial\tilde{\rho}}{\partial\tilde{r}}$	0.6 pt
B-9	$\ddot{\epsilon}=-\frac{4\pi G\rho_c}{3}(3\gamma-4)\epsilon$ (0.4 pt) $\gamma_{min}=\frac{4}{3}$ (0.1 pt) $\omega=\sqrt{\frac{4\pi G\rho_c}{3}(3\gamma-4)}$ (0.1 pt)	0.6 pt