The Greenhouse Effect

In 2021, Syukuro Manabe and Klaus Hasselmann were awarded half of the Nobel Prize in Physics for their work in modeling Earth's climate and accurately predicting the global warming caused by human industrial activities. In this problem, we will examine a simple model of global warming due to the greenhouse effect. The greenhouse gases alter the optical properties of the Earth's atmosphere in transmitting or absorbing Earth's infrared radiation, resulting in a rise in the average temperature of the planet.

All objects, at different temperatures, emit thermal radiation. The quantity $u(\lambda, T)d\lambda$ indicates the thermal radiative power per unit area of an object at temperature T between the wavelengths λ and $\lambda + d\lambda$. According to Planck's theory of blackbody radiation, we have

$$u(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{\exp(\frac{hc}{\lambda k_\mathrm{B} T}) - 1}, \quad (1)$$

in which $hc = 1.24 \times 10^3 \ \mathrm{eV \cdot nm}$ and $k_\mathrm{B} = 8.62 \times 10^{-5} \ \mathrm{eV/K}$. The wavelength corresponding to the maximum of $u(\lambda, T)$ comes from the relation $\lambda_\mathrm{max} T = b$ (Wien's displacement law). Indeed, using equation (1), it can be shown that $b = \frac{hc}{x_\mathrm{m} k_\mathrm{B}}$, where the dimensionless quantity x_m is the non-trivial root of an equation of the form $f(x) = 0$; you are asked to find the function $f(x)$ in one of the following tasks. Total radiative power per unit area of a blackbody in all wavelengths is given by the Stephan-Boltzmann law as $U(T) = \sigma T^4$ where $\sigma = 5.67 \times 10^{-8} \ \mathrm{W/m^2K^4}$. Moreover, according to Kirchhoff's law of radiation, at thermal equilibrium a body absorbing a certain fraction of the incident radiation at a specific wavelength, will radiate the same fraction of the blackbody radiation at that same wavelength.

Throughout this problem assume that the Sun is a blackbody at its average surface temperature of $T_\mathrm{S} = 5.77 \times 10^3 \ \mathrm{K}$. The Sun's radius is $R_\mathrm{S} = 6.96 \times 10^8 \ \mathrm{m}$ and the average distance between the Earth and the Sun is $d = 1.50 \times 10^{11} \ \mathrm{m}$. We denote by $\tilde{u}_\mathrm{S}(\lambda)$, the spectral solar power radiated into a unit area of the Earth normal to the direction of radiation. The integral of this quantity over all wavelengths, i.e. $S_0 = \int \tilde{u}_\mathrm{S}(\lambda) d\lambda$, is called the solar constant.

In this problem assume that the Earth is in thermal equilibrium and has the same temperature at all points on its surface. In all parts of the problem, express the desired quantity in parametric form in terms of the data given in the problem and then find its numerical value accurate to three significant figures. The required units are indicated on the answer sheet.

A. Earth as a Blackbody

In this part, consider the Earth's surface as a blackbody and neglect the Earth's atmosphere.

A-1	Find the solar constant, S_0.	0.6 pt

A-2	Find the Earth's temperature, T_E .	0.6 pt

A-3	Find the function $f(x)$.	0.4 pt

A-4	Calculate the numerical value of x_m , and from this value x_m , find the value of b.	0.4 pt

A-5	Find λ_{max} for the Sun and the Earth.	0.2 pt

In figure 1 the functions $\gamma\tilde{u}_S(\lambda)$ and $u(\lambda, T_E)$ are plotted versus λ, where γ is a dimensionless coefficient to rescale $\tilde{u}_S(\lambda)$ such that the values of the two peaks coincide.

A-6	Determine γ.	0.8 pt

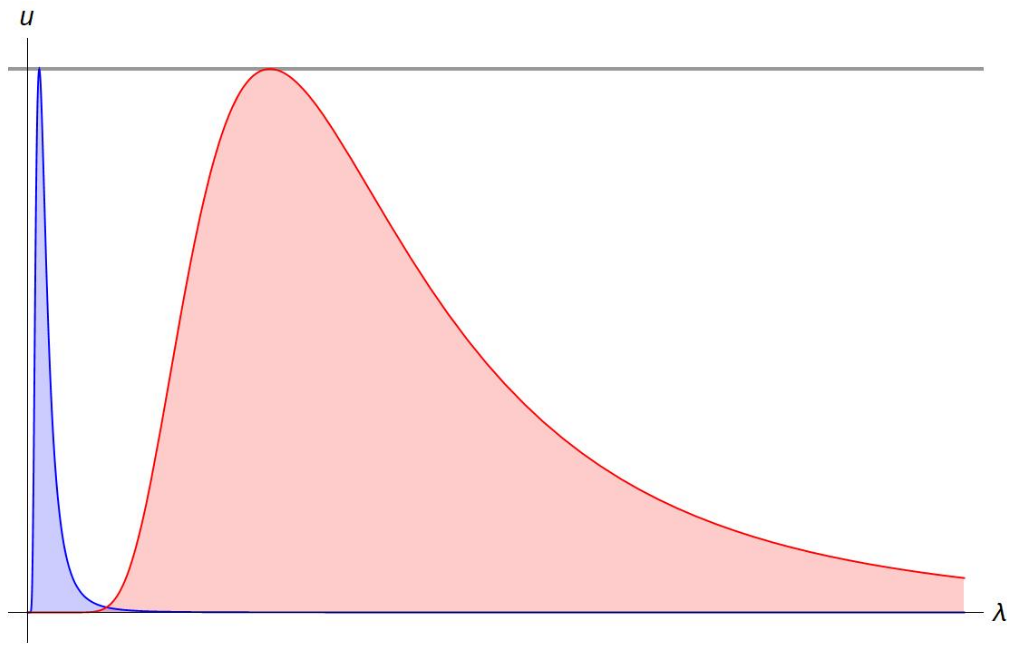

Figure 1 - The plot of $u(\lambda, T_{\mathrm{E}})$ *(red) and* $\gamma \tilde{u}_{\mathrm{S}}(\lambda)$ *(blue) versus* λ

B. The Greenhouse Effect

In this part, we introduce a simple model in which the Earth's atmosphere is modeled as a thin layer at a small distance above the Earth's surface so that the difference between the area of the atmosphere's layer and the area of the Earth's surface can be neglected (see figure 2). In what follows assume that the major part of the thermal radiation from the Earth and the Sun are emitted at wavelengths near the $\lambda_{\max}$ for each one. Also assume that the "atmosphere layer" reflects a fraction $r_{\mathrm{A}} = 0.255$ of the **visible-ultraviolet** radiation incident from above or below, and completely transmits the rest. Assume that the atmosphere does not reflect any part of the **infrared** radiation, however, it absorbs a fraction ε of the **infrared** radiation and transmits the rest. This behavior, known as the greenhouse effect, changes the average temperature of the Earth. The Earth's surface, on the other hand, reflects a fraction r_{E} of the **visible-ultraviolet** radiation and absorbs the rest of this radiation and all the **infrared** radiation.

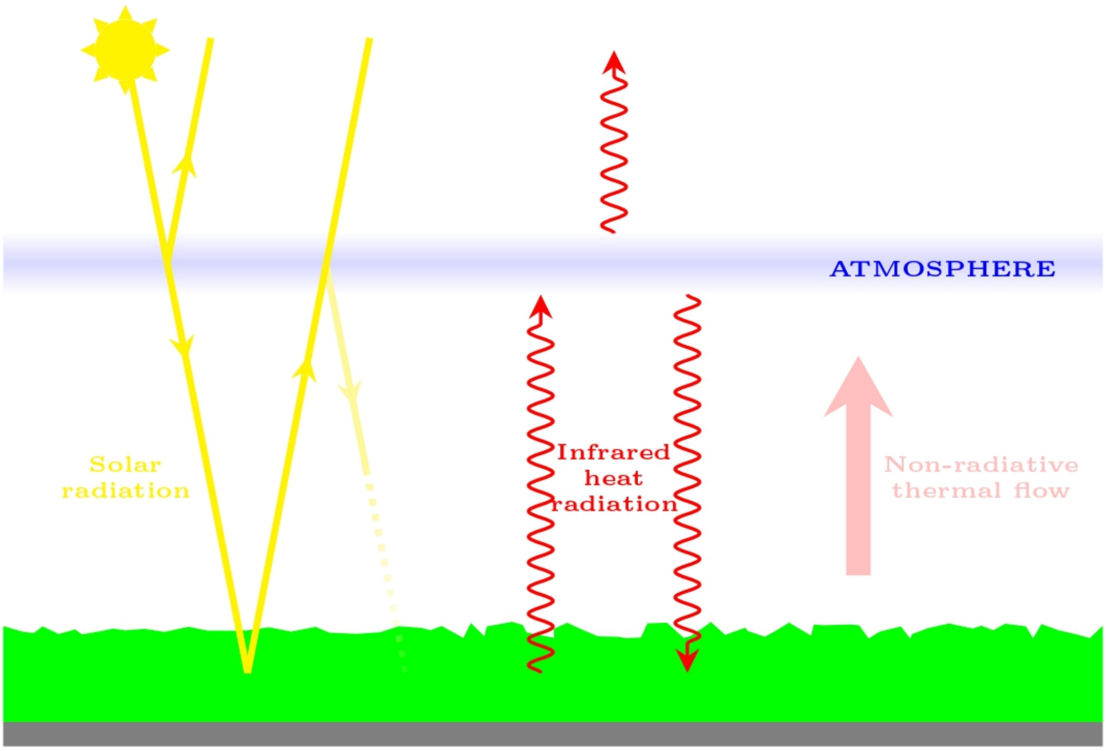

Figure 2 - Thermal flows between the Earth and the atmosphere

B-1	Assume that $\varepsilon = 1$ and $r_{\mathrm{E}} = 0$, and calculate the Earth's temperature T_{E} and the atmosphere's temperature T_{A}.	1.0 pt

Now assume that $r_{\mathrm{E}} \neq 0$. In this case, the combined system of "Earth + atmosphere" reflects a different fraction of the solar radiation, called "albedo" and denoted by α.

B-2	Determine the albedo, α, in terms of r_{E} and r_{A}. Then calculate its numerical value assuming $r_{\mathrm{E}} = 0.102$ (and $r_{\mathrm{A}} = 0.255$).	1.6 pt

B-3	a) Express the Earth's temperature in terms of σ, α, S_0, and ε. b) Using the given data and the calculated albedo, find the numerical value of ε which leads to the current average temperature of $T_\mathrm{E} = 288\ \mathrm{K}$ for the Earth.	1.0 pt

B-4	Find $\frac{dT_\mathrm{E}}{d\varepsilon}$ and determine by how much the Earth's temperature increases if ε increases by one percent.	0.8 pt

Assume $T_\mathrm{A} = 245\ \mathrm{K}$ and $T_\mathrm{E} = 288\ \mathrm{K}$. These values come from real data and may differ from the results which you have obtained in the previous tasks. Now suppose that a non-radiative (e. g. convective) thermal flow $J_\mathrm{NR} = k(T_\mathrm{E} - T_\mathrm{A})$ is maintained from the Earth to the atmosphere, where k is a constant. The quantity, J_NR, is the transmitted power per unit area.

B-5	Calculate ε and k in terms of T_E, T_A, σ, α, and S_0.	1.6 pt

B-6	a) Differentiating the equations obtained in part B-5 with respect to ε, find the two algebraic equations satisfied by $\frac{dT_\mathrm{A}}{d\varepsilon}$ and $\frac{dT_\mathrm{E}}{d\varepsilon}$. b) Use these equations to find the numerical value of change in the Earth's temperature as a result of a one percent increase in the value of ε.	1.0 pt

A. Earth as a blackbody

A-1	$S_0 =$	$S_0 =$ $\quad$ W/m^2	0.6 pt
A-2	$T_E =$	$T_E =$ $\quad$ K	0.6 pt
A-3	$f(x) =$		0.4 pt
A-4	$x_m =$	$b =$ $\quad$ nm. K	0.4 pt
A-5	$\lambda_{max}^S =$ $\quad$ nm	$\lambda_{max}^E =$ $\quad$ nm	0.2 pt
A-6	$\gamma =$	γ(numerical) =	0.8 pt

B. The Greenhouse Effect

B-1	$T_E =$ $T_A =$	$T_E =$ $\quad$ K $T_A =$ $\quad$ K	1.0 pt
B-2	$\alpha =$	α(numerical) =	1.6 pt
B-3	$T_E =$	ϵ(numerical) =	1.0 pt
B-4	$\frac{dT_E}{d\epsilon} =$	$\delta T_E =$ $\quad$ K	0.8 pt
B-5	$\epsilon =$ $k =$	ϵ (numerical) = $k =$ $\quad$ W/m^2K	1.6 pt

B-6	(a)	(b) $\delta T_E =$ K	1.0 pt

Trapping Ions and Cooling Atoms

In recent decades, trapping and cooling atoms and ions has been a fascinating topic for physicists, with several Nobel prizes awarded for work in this area. In the first part of this question, we will explore a technique for trapping ions, known as the "Paul trap". Wolfgang Paul and Hans Dehmelt received one half of the 1989 Nobel Prize in Physics for this work. Next, we investigate the Doppler cooling technique, one of the works cited in the press release for the 1997 Nobel Prize in Physics awarded to Steven Chu, Claude Cohen-Tannoudji, and William Daniel Phillips "for developments of methods to cool and trap atoms with laser light".

A. The Paul Trap

It is known that with electrostatic fields, it is not possible to create a stable equilibrium for a charged particle. Therefore, creating a stable equilibrium point for ions requires more sophisticated techniques. The Paul trap is one of these techniques.

Consider a ring of charge with a radius R and a uniform positive linear charge density λ. A positive point charge Q with mass m is placed at the center of the ring.

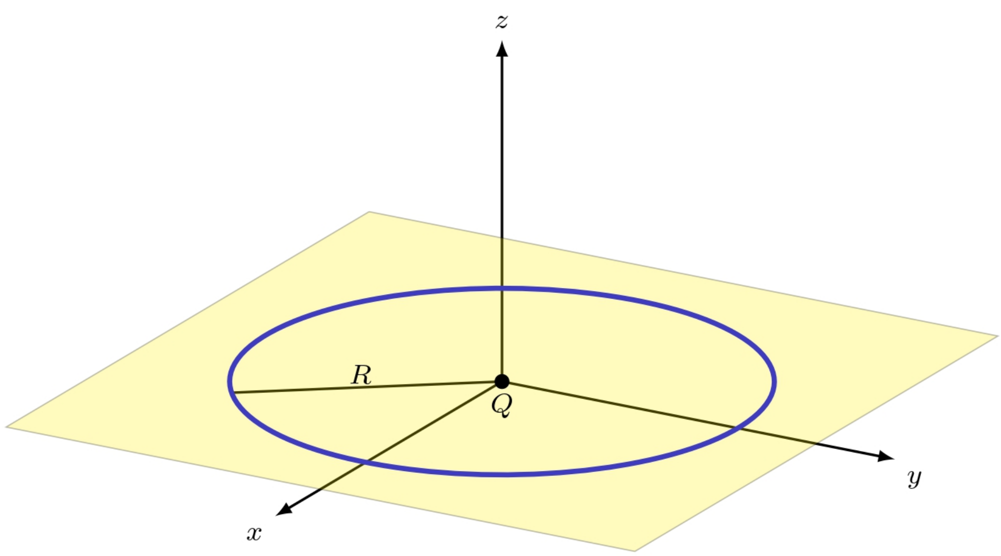

Figure 1 - A positively charged ring with a uniform linear charge density λ and radius R: the origin of the coordinate system is at the center of the ring.

A-1	a) In cartesian coordinates (x, y, z), obtain the electric field due to the charged ring in the vicinity of the ring's center to the first order in x/R, y/R, and z/R. b) Find the angular frequency of small oscillations of the charged particle around the center of the ring in the directions for which a stable equilibrium exists.	1.5 pt

In order to trap the charge Q fully, we would like to apply alternating fields to produce a dynamic equilibrium. Assume that the charge density is $\lambda = \lambda_0 + u \cos \Omega t$ in which λ_0, u, and Ω are adjustable. We shall ignore radiative effects. Then the equation of motion for small displacements from the center of the ring, along the direction perpendicular to the plane of the ring will turn out to be:

$$\ddot{z} = (+k^2 + a\Omega^2 \cos \Omega t)\, z \qquad (1)$$

A-2	Write a and k in terms of the known parameters.	0.4 pt

We would like to obtain an approximate solution to Equation (1) by making the following simplifying assumptions: $a \ll 1$, $\Omega \gg k$, and $a\Omega^2 \gg k^2$. With these assumptions, it can be shown that the solution of this equation can be split into two parts: $z(t) = p(t) + q(t)$, where $p(t)$ is a slowly varying component and $q(t)$ is a <u>small-amplitude</u> rapidly-varying component with a <u>mean value of zero</u>. In other words, $p(t)$ may be assumed constant over a few oscillations of $q(t)$ (see Figure 2).

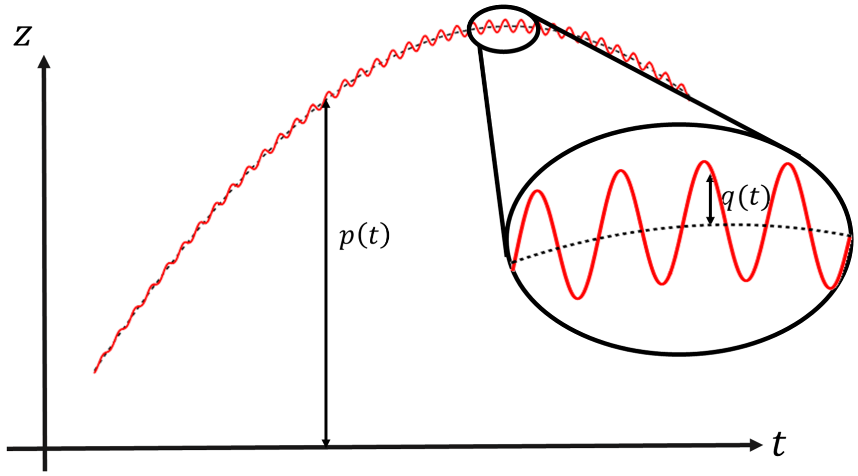

Figure 2 - A typical solution for the equation of motion of the charged particle: $p(t)$ gives the overall motion, and $q(t)$ represents small oscillations around this trajectory. The ellipse on the right is a magnification of a part of this trajectory.

A-3	a) Using the approximations stated above, find the equation of motion for $q(t)$ in terms of a, Ω, and p. b) Find the solution of this equation by considering appropriate initial conditions corresponding to the required properties of this function.	1.8 pt

A-4	a) Using the mean effect of the rapidly varying component and obtain an effective equation of motion for $p(t)$. b) Investigate the stability of the equilibrium point and find the condition for a stable equilibrium.	1.5 pt

Assume that $\lambda_0 = 8 \times 10^{-9}\ \mathrm{C/m}$ and $R = 10\ \mathrm{cm}$. We would like to use this device to trap a singly ionized atom 100 times heavier than a hydrogen atom.

A-5	Calculate k. Assume $a = 0.04$ and estimate the smallest frequency required to stabilize the motion of this ion. Use the data given at the end of the question.	0.4 pt

B. Doppler Cooling

It may be necessary to cool a trapped atom or ion. Assume that a trapped atom of mass m, has two energy levels with an energy difference of $E_0 = \hbar\omega_{\mathrm{A}}$. Electrons in the lower level may absorb a photon and jump to the higher level, but after a period τ they will return to the lower level and emit a photon with a frequency predominantly within $[\omega_{\mathrm{A}} - \mathit{\Gamma}, \omega_{\mathrm{A}} + \mathit{\Gamma}]$.

B-1	Use the Heisenberg's uncertainty principle to find $\mathit{\Gamma}$.	0.5 pt

With a similar reasoning, when we shine a laser light on the trapped atom, if the angular frequency of the laser, ω_{L}, falls in the interval $[\omega_{\mathrm{A}} - \mathit{\Gamma}, \omega_{\mathrm{A}} + \mathit{\Gamma}]$, the atom may absorb the photon. Assume that the frequency ω_{L} of the laser light is slightly lower than ω_{A}. For a particular device, the rate of photon absorption by an atom in the reference frame of the atom is given in Figure 3. The absorbed photon is then re-emitted in a random direction. To make things simple, we consider the problem in one dimension, i.e. we assume that the atoms can only move in the x-direction and the laser light shines on them both from the left and from the right. In the atom's reference frame, the light has a higher or lower frequency due to the motion of the atoms. Since the velocity v of the atoms is very small, we only include terms of order v/c and ignore all the higher-order terms. Moreover, we have $m \gg \hbar\omega_{\mathrm{A}}/c^2$ so that the velocity of the atom nearly does not change after absorbing the photon. Also, the change in frequency due to the Doppler effect is so small compared to $\omega_{\mathrm{A}} - \omega_{\mathrm{L}}$, that the function for s in the diagram of Figure 3 may be approximated by the following linear function:

$$s(\omega) = s_{\mathrm{L}} + \alpha(\omega - \omega_{\mathrm{L}})$$

where s is the number of absorbed photons per unit of time, s_{L} is the value of s for $\omega = \omega_{\mathrm{L}}$, and α is the slope of the line tangent to the curve at ω_{L}. The frequency of the re-emitted photon is almost equal to the frequency of the incident photon, but it is emitted with equal probability in the positive or negative x-direction. In fact, up to the order considered here the two frequencies are identical. Note that we are considering the whole process in the atom's reference frame.

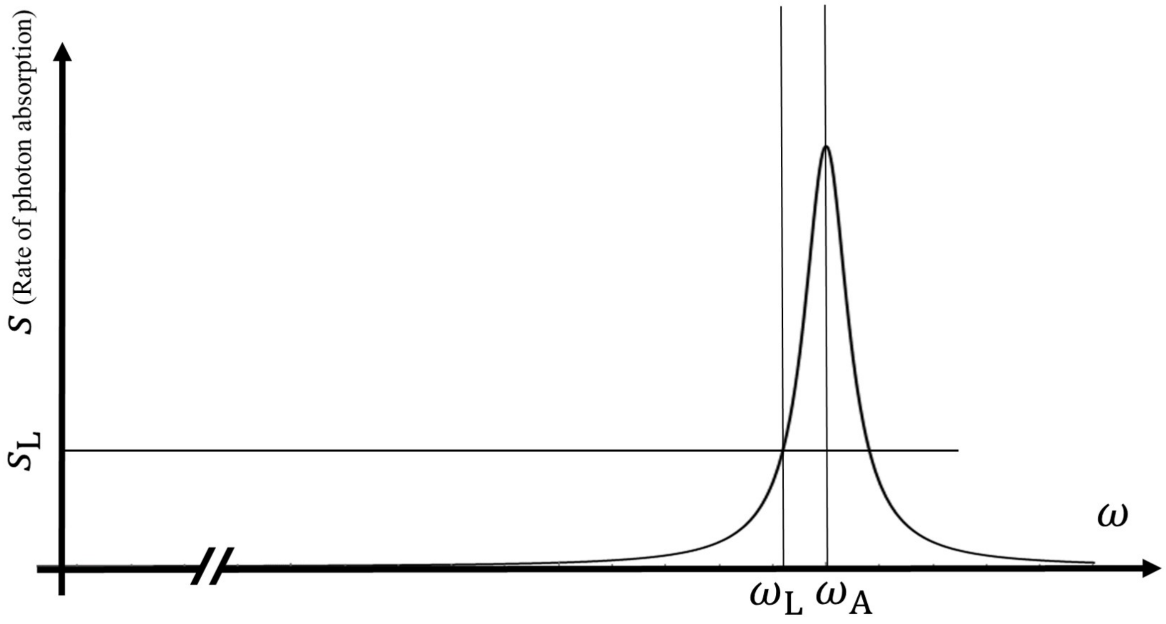

Figure 3 - The rate of photon absorption as a function of the frequency for a particular trap: the frequency corresponding to the energy difference between the two atomic levels is indicated by ω_A and the slighlty smaller frequency of the laser is indicated by ω_L.

B-2	a) Assume that the trapped atom is moving with a velocity, $v = v_x$ in the lab frame. In the frame of reference of the atom, calculate the collision rate of the photons, incident from each of the two directions, with the atoms (denoted by s_+ and s_-) and the rate of absorption of momentum in each direction (denoted by π_+ and π_-). b) Determine the effective force on the atom as a function of v, $k_L = \omega_L/c$, $\hbar$, and α, in the reference frame of the laboratory. Assume $s_L \ll \alpha\omega_L$,	1.7 pt

We would like to find the lowest temperature that can be achieved using this technique. Assume that the velocity of a particular atom has been reduced to zero exactly, and at this very moment it absorbs a photon (incident from any of the two directions), and re-emits the photon randomly in any of the two directions, with almost the same frequency. Assume that this process happens once every τ units of time.

B-3	Considering the momentum of the atom after such a process for the two possible outcomes, calculate the average power absorbed by the atom.	1.0 pt

B-4	Consider the force calculated in Task B-2 and calculate the output power. Then, calculate the average value of v^2 at equilibrium. Using your knowledge of the kinetic theory of gases estimate the temperature of the atoms.	0.8 pt

B-5	Estimate this temperature, for an atom 100 times heavier than a hydrogen atom. Assume that $\omega_L = 2 \times 10^{16}$ rad /s, $\tau = 5 \times 10^{-9}$ s , and $\alpha = 4$.	0.4 pt

mass of hydrogen atom: $m_H = 1.674 \times 10^{-27}$ kg
charge of an electron: $e = 1.602 \times 10^{-19}$ C
permittivity of free space: $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m
Boltzmann constant: $k_B = 1.381 \times 10^{-23}$ J/K
Planck constant: $\hbar = 1.055 \times 10^{-34}$ J.s

A: Paul Trap

A.1	(a) $\vec{E}(x,y,z) =$	(b)	1.5 pt

A.2	$k =$	$a =$	0.4 pt

A.3	(a)	(b)	1.8 pt

A.4	(a)	(b)	1.5 pt

A.5	$k =$ rad/s	$\Omega_{\min} \simeq$ rad/s	0.4 pt

B: Doppler Cooling

B.1	$\Gamma =$	0.5 pt

B.2	$s_- =$ $s_+ =$	$\pi_- =$ $\pi_+ =$	1.7 pt
	$F =$		

B.3	$P_{\text{in}} =$	1.0 pt

B.4	$P_{\text{out}} =$	$\overline{v^2} =$	$T =$	0.8 pt

B.5	$T =$ K	0.4 pt

Black Widow Pulsar

A significant number of the observed stars are binaries. One or both of the stars may be neutron stars rotating with a high angular velocity and emitting electromagnetic waves; such stars are called pulsars. Sometimes a companion star is an expansive mass of gas that gradually falls down onto the neutron star and causes its mass to increase (Figure 1-a). In this way, a neutron star gradually swallows up a portion of the mass of its companion star. For this reason, the neutron star has been compared to a black widow (or redback spider), a female spider which eats its mate after mating. The heating of the gas falling down onto the black widow generates radiation which can be observed. The heaviest neutron stars often are black widows and they serve as natural laboratories for testing fundamental physics. Figure 1-b shows the picture of the companion of the neutron star PSR J2215+5135, taken by the 3.4-meter optical telescope of the Iranian National Observatory. No neutron star can be seen in this image and the observed light is due to its companion.

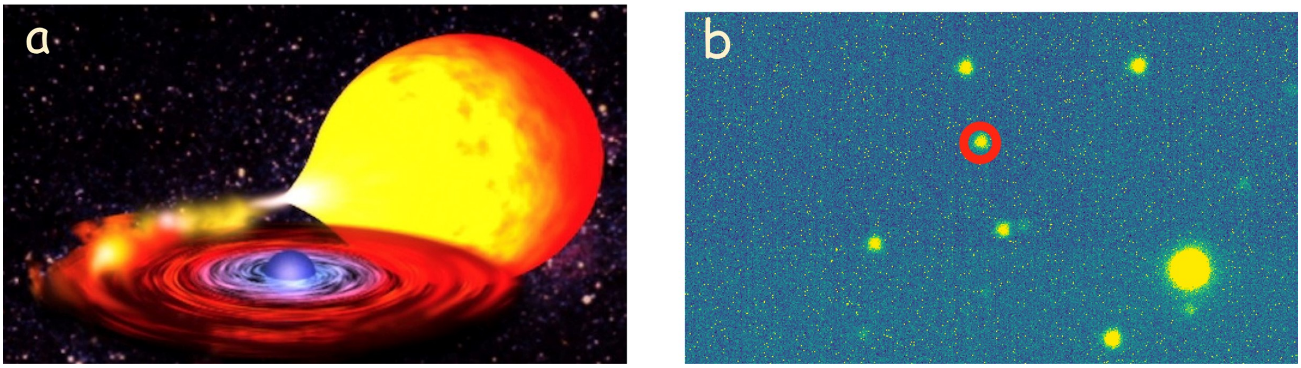

Figure 1 - (a) The falling gases of the companion star onto the neutron star - (b) The companion of the neutron star PSR J2215+5135

A. A Binary System

Consider a simple model in which the black widow and its companion star, are represented by two point masses M_1 and M_2 moving on a circular orbit around their center of mass. To investigate the dynamics of this system, consider a rotating coordinate system in which the two bodies are stationary. Take the center of mass to be the origin of the coordinate system. Assume that the two point masses lie on the x-axis on both sides of the origin at a distance a from each other, and that M_1 lies on the negative x-axis. At an arbitrary point (x, y) in the plane of motion, the effective potential $\varphi(x, y)$ for a unit test mass is the sum of the gravitational potentials of the two point masses plus the centrifugal potential.

A-1	Write $\varphi(x, y)$ in terms of M_1, M_2, G, and a.	1.0 pt

A-2	Assuming $M_1 > M_2$, plot the function $\varphi(x, 0)$ qualitatively.	0.7 pt

Suppose (just for task A-3) $M_2 = M_1/3$ and assume that M_2 is surrounded by a rarefied gas of very low density. The mass of this gas is insignificant and we ignore its gravitational effects. If the size of this gas envelope becomes greater than a specific limit, the gas will overflow onto M_1. Suppose the overflow occurs through $x = x_0$ on the x-axis.

A-3	Find the numerical value of $\frac{x_0}{a}$, up to two significant figures. You may use the calculator.	0.5 pt

Take the rotational period of the stars around their center of mass to be P. Assume that mass flows from M_2 to M_1 at a very small rate of $dM_1/dt = \beta$. This rate is so small that in each period of rotation, the distance between the two stars can be assumed to be constant. However, after a long period of time, the distance between the two stars changes, while the motion remains circular.

A-4	Calculate the rate of change of a and P in terms of β, M_1 , M_2 , G, and a.	0.6 pt

The gas separated from M_2 forms a disk rotating around M_1 and heats up due to friction (Figure 1-a). As the gas loses energy, it spirals inward toward M_1 and finally falls onto it. In the steady state, the mass flows at the constant rate of β, from M_2 to the disc and from the disc onto M_1. At the same time, the heated disk emits thermal radiation as a blackbody. This disk forms very close to the neutron star so the gravitational pull of the M_2 star can be ignored for the analysis of the disk's motion. Also, ignore the heat capacity of the gas.

A-5	Determine the temperature of the disc at distance r from the center of the star M_1 in terms of β, M_1, G, and σ (Stefan-Boltzmann constant).	1.0 pt

In the binary system PSR J2215+5135, the mass of the neutron star is $M_{\mathrm{NS}} = 2.27\ M_\odot$ and the mass of its companion star is $M_{\mathrm{S}} = 0.33\ M_\odot$, where $M_\odot = 1.98 \times 10^{30}\ \mathrm{kg}$ is the mass of the Sun. The rotational period is $P = 4.14\ \mathrm{hr}$, and the Stefan-Boltzmann constant is $\sigma = 5.67 \times 10^{-8}\ \mathrm{W/m^2K^4}$, and the gravitational constant is $G = 6.67 \times 10^{-11}\ \mathrm{m^3/\,kgs^2}$. Assume that the mass flow rate to the neutron star is $\beta = \dot{M}_{\mathrm{NS}} = 9 \times 10^{-10}\ M_\odot\,\mathrm{yr}^{-1}$.

A-6	Calculate the temperature of the disc at the radius $r = \frac{a}{10}$ in kelvins.	0.5 pt

Assume that after a sudden explosion, the M_1 star ejects a part of its mass out of the binary system at a very high speed, and its mass becomes M_1'. Take the magnitude of the velocity of M_1' relative to M_2 to be v' after the explosion.

A-7	Determine the maximum value of v', in terms of M_1', M_2, G, and a, that allows the new binary system to stay bounded. Assuming that the explosion is isotropic, what is the minimum value of M_1' for the binary system to remain bounded?	0.7 pt

B. Analysis of the Stability of a Star

In this part we study the stability of a single star. Consider a star containing a specific kind of matter with the equation of state $p = K\rho^\gamma$ where K and γ are constants. Let $p(r)$ and $\rho(r)$ be the pressure and density at a distance r from the center of the star, respectively. The pressure and density at the center of the star are p_c and ρ_c, respectively. In all tasks of the part B, take all outward vectors to be positive.

B-1	Determine the gravitational acceleration $g(r)$ near the center of the star in terms of r and the constants G and ρ_c.	0.2 pt
B-2	Derive a (differential) equation for determining $\rho(r)$ at equilibrium, and write it in the following form: $\frac{d}{dr}[h_1(\rho, r)\frac{d\rho}{dr}] + h_2(r)\rho = 0$. Find the functions h_1 and h_2.	0.6 pt
B-3	Construct a quantity r_0 of the form $r_0 = G^l {p_c}^m {\rho_c}^n$ with the dimension of length.	0.4 pt
B-4	Rewrite the (differential) equation of task B-2 in the following form: $$\frac{d}{dx}[A_1(u,x)\frac{du}{dx}] + A_2(x)u(x) = 0,$$ where $x = \frac{r}{r_0}$ and $u = \frac{\rho}{\rho_c}$. Find the functions $A_1(u,x)$ and $A_2(x)$.	0.3 pt
B-5	For $\gamma = 2$ one finds $u(x) = \frac{f(x)}{x}$. Determine $f(x)$.	0.6 pt

Assume that for a particular star $\frac{du}{dx}$, as a function of x, is given by the curve given in Figure 2.

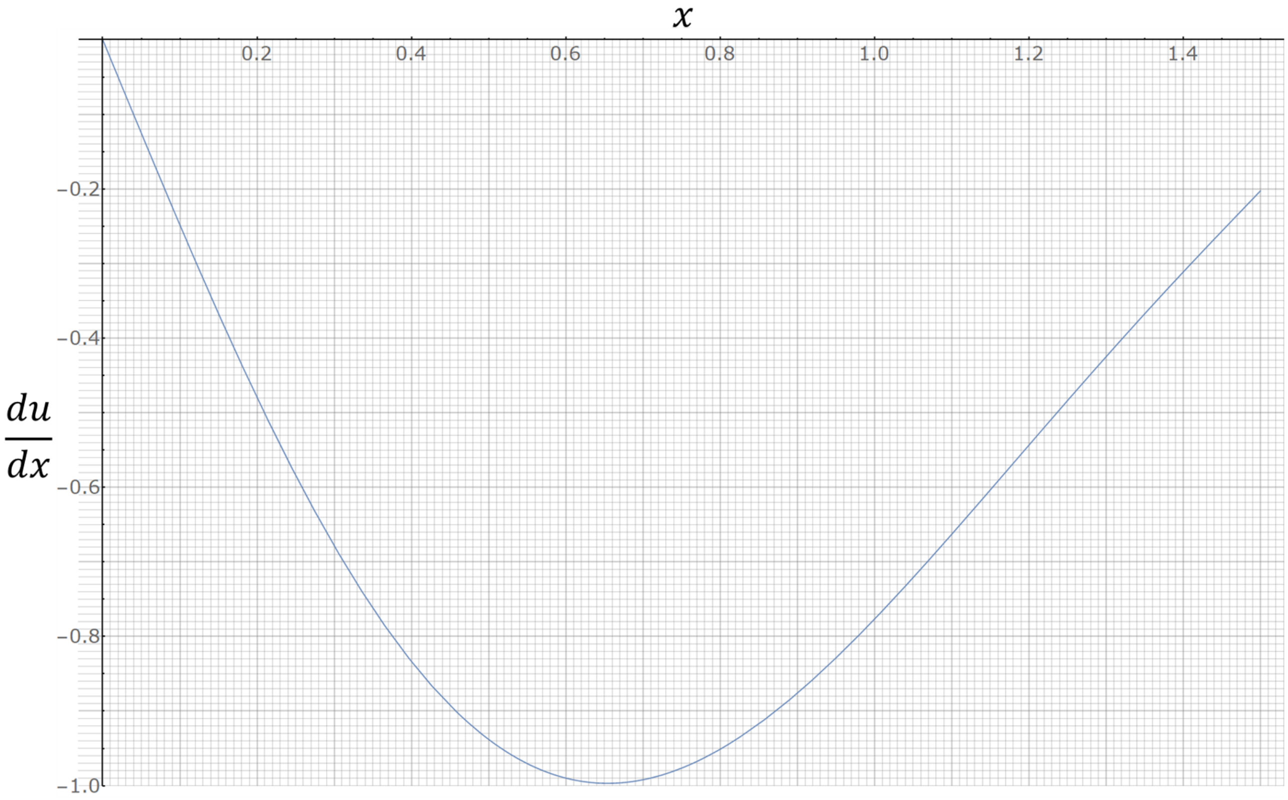

Figure 2 - The plot of $\frac{du}{dx}$

B-6	Use the behavior of the curve in Figure 2, in the vicinity of the point $x = 0$, to find γ up to 3 significant figures. Use the given ruler if necessary.	0.8 pt

To analyze the stability of the system, we assume that the star deviates slightly from its equilibrium state: we assume that the spherical shell, which was in equilibrium at radius r, now has a radius $\tilde{r}$, similarly the parameters g, p, and ρ have changed to $\tilde{g}$, $\tilde{p}$, and $\tilde{\rho}$ respectively. For convenience, we shall only consider small r's near the center of the star, for which we can assume that $\tilde{r} = r(1 + \varepsilon(t))$, where $\varepsilon(t) \ll 1$.

B-7	Find $\tilde{\rho}$ and $\tilde{g}$ in terms of ρ and g to the first order in ε.	0.9 pt

B-8	Using Newton's equation of motion for the spherical layer with the equilibrium radius of r find $\frac{d^2\tilde{r}}{dt^2}$ in terms of $\tilde{g}$, $\tilde{\rho}$, K, γ, and $\frac{\partial \tilde{p}}{\partial \tilde{r}}$ (By $\frac{\partial \tilde{p}}{\partial \tilde{r}}$ we mean derivative of $\tilde{p}$ with respect to $\tilde{r}$ at constant t.)	0.6 pt

B-9	Obtain $\frac{d^2\varepsilon}{dt^2}$ in terms of ε and the constants given in the problem. Find the minimum value of γ for a stable equilibrium, and find the oscillation's angular frequency of the star.	0.6 pt

A. A Binary System

A-1	$\Phi(x, y) =$	1.0 pt

A-2		0.7 pt

A-3	$\frac{x_0}{a} =$	0.5 pt

A-4	$\dot{a} =$	$\dot{P} =$	0.6 pt

A-5	$T =$	1.0 pt

A-6	$T =$ K	0.5 pt

A-7	$v'_{max} =$	$M'_{1\,min} =$	0.7 pt

B. Analysis of the stability of a star

B-1	$g =$			0.2 pt
B-2	$h_1(\rho, r) =$	$h_2(r) =$		0.6 pt
B-3	$r_0 =$			0.4 pt
B-4	$j_1(u, x) =$	$j_2(x) =$		0.3 pt
B-5	$f(x) =$			0.6 pt
B-6	$\gamma =$			0.8 pt
B-7	$\tilde{g} \simeq$	$\tilde{\rho} \simeq$		0.9 pt
B-8	$\frac{d^2\tilde{r}}{dt^2} =$			0.6 pt
B-9	$\ddot{\epsilon} =$	$\gamma_{min} =$	$\omega =$	0.6 pt