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Solutions to Theory Problem 1

LIGO-GW150914

(V. Cardoso, C. Herdeiro)

July 15, 2018

v6.0

GW150914 (10 points)

Part A. Newtonian (conservative) orbits (3.0 points)

A.1 Apply Newton's law to mass M_1 :

$$M_1 \frac{d^2 \vec{r}_1}{dt^2} = G \frac{M_1 M_2}{|\vec{r}_2 - \vec{r}_1|^2} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}. \quad (1)$$

Use, from eq. (1) of the question sheet

$$\vec{r}_2 = -\frac{M_1}{M_2} \vec{r}_1, \quad (2)$$

in eq. (1) above, to obtain

$$\frac{d^2 \vec{r}_1}{dt^2} = -\frac{GM_2^3}{(M_1 + M_2)^2 r_1^2} \frac{\vec{r}_1}{r_1}. \quad (3)$$

A.1

1.0pt

$$n = 3, \quad \alpha = \frac{GM_2^3}{(M_1 + M_2)^2}.$$

A.2 The total energy of the system is the sum of the two kinetic energies plus the gravitational potential energy. For circular motions, the linear velocity of each of the masses reads

$$|\vec{v}_1| = r_1 \Omega, \quad |\vec{v}_2| = r_2 \Omega, \quad (4)$$

Thus, the total energy is

$$E = \frac{1}{2}(M_1 r_1^2 + M_2 r_2^2) \Omega^2 - \frac{GM_1 M_2}{L}, \quad (5)$$

Now,

$$(M_1 r_1 - M_2 r_2)^2 = 0 \quad \Rightarrow \quad M_1 r_1^2 + M_2 r_2^2 = \mu L^2. \quad (6)$$

Thus,

$$E = \frac{1}{2} \mu L^2 \Omega^2 - G \frac{M \mu}{L}. \quad (7)$$

A.2

1.0pt

$$A(\mu, \Omega, L) = \frac{1}{2} \mu L^2 \Omega^2.$$

A.3 Energy (3) of the question sheet can be interpreted as describing a system of a mass μ in a circular orbit with angular velocity Ω , radius L , around a mass M (at rest). Equating the gravitational acceleration to the centripetal acceleration:

$$G \frac{M}{L^2} = \Omega^2 L. \quad (8)$$

This is indeed Kepler's third law (for circular orbits). Then, from (7),

$$E = -\frac{1}{2} G \frac{M \mu}{L}. \quad (9)$$

A.3

1.0pt

$$\beta = -\frac{1}{2}.$$

Part B - Introducing relativistic dissipation (7.0 points)

B.1 Some simple trigonometry for the x, y motion of the masses (in an appropriate Cartesian system) yields:

$$(x_1, y_1) = r_1 (\cos(\Omega t), \sin(\Omega t)), \quad (x_2, y_2) = -r_2 (\cos(\Omega t), \sin(\Omega t)). \quad (10)$$

Then,

$$Q_{ij} = \frac{M_1 r_1^2 + M_2 r_2^2}{2} \begin{pmatrix} \frac{4}{3} \cos^2(\Omega t) - \frac{2}{3} \sin^2(\Omega t) & 2 \sin(\Omega t) \cos(\Omega t) & 0 \\ 2 \sin(\Omega t) \cos(\Omega t) & \frac{4}{3} \sin^2(\Omega t) - \frac{2}{3} \cos^2(\Omega t) & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}, \quad (11)$$

or, using some simple trigonometry and (6),

$$Q_{ij} = \frac{\mu L^2}{2} \begin{pmatrix} \frac{1}{3} + \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & \frac{1}{3} - \cos 2\Omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}. \quad (12)$$

B.1

1.0pt

$$k = 2\Omega, \quad a_1 = a_2 = \frac{1}{3}, a_3 = -\frac{2}{3}, \quad b_1 = 1, b_2 = -1, b_3 = 0, c_{12} = c_{21} = 1, c_{ij} \stackrel{\text{otherwise}}{=} 0.$$

B.2 First take the derivatives:

$$\frac{d^3 Q_{ij}}{dt^3} = 4\Omega^3 \mu L^2 \begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (13)$$

Then perform the sum:

$$\frac{dE}{dt} = \frac{G}{5c^5} (4\Omega^3 \mu L^2)^2 [2 \sin^2(2\Omega t) + 2 \cos^2(2\Omega t)] = \frac{32}{5} \frac{G}{c^5} \mu^2 L^4 \Omega^6. \quad (14)$$

B.2

1.0pt

$$\xi = \frac{32}{5}.$$

B.3 Now we assume a sequence of Keplerian orbits, with decreasing energy, which is being taken from the system by the GWs.

First, from (9), differentiating with respect to time,

$$\frac{dE}{dt} = \frac{GM\mu}{2L^2} \frac{dL}{dt}, \quad (15)$$

Since this loss of energy is due to GWs, we equate it with (minus) the luminosity of GWs, given by (14)

$$\frac{GM\mu}{2L^2} \frac{dL}{dt} = -\frac{32}{5} \frac{G}{c^5} \mu^2 L^4 \Omega^6. \quad (16)$$

We can eliminate the L and dL/dt dependence in this equation in terms of Ω and $d\Omega/dt$, by using Kepler's third law (8), which relates:

$$L^3 = G \frac{M}{\Omega^2}, \quad \frac{dL}{dt} = -\frac{2}{3} \frac{L}{\Omega} \frac{d\Omega}{dt}. \quad (17)$$

Substituting in (16), we obtain:

$$\left(\frac{d\Omega}{dt}\right)^3 = \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} G^5 \mu^3 M^2 \equiv \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} (GM_c)^5. \quad (18)$$

B.3

1.0pt

$$M_c = (\mu^3 M^2)^{1/5}.$$

B.4 Angular and cycle frequencies are related as $\Omega = 2\pi f$. From the information provided above: *GWs have a frequency which is twice as large as the orbital frequency*, we have

$$\frac{\Omega}{2\pi} = \frac{f_{\text{GW}}}{2}. \quad (19)$$

Formula (10) of the question sheet has the form

$$\frac{d\Omega}{dt} = \chi \Omega^{11/3}, \quad \chi \equiv \frac{96 (GM_c)^{5/3}}{5 c^5}. \quad (20)$$

Thus, from (11) of the question sheet

$$\Omega(t)^{-8/3} = \frac{8}{3} \chi (t_0 - t), \quad (21)$$

or, using (20) and the definition of χ gives

$$f_{\text{GW}}^{-8/3}(t) = \frac{(8\pi)^{8/3}}{5} \left(\frac{GM_c}{c^3}\right)^{5/3} (t_0 - t). \quad (22)$$

B.4

2.0pt

$$p = 1.$$

B.5 From the figure, we consider the two Δt 's as half periods. Thus, the (cycle) GW frequency is $f_{\text{GW}} = 1/(2\Delta t)$. Then, the four given points allow us to compute the frequency at the mean time of the two intervals as

	$t_{\overline{AB}}$	$t_{\overline{CD}}$
t (s)	0.0045	0.037
f_{GW} (Hz)	$(2 \times 0.009)^{-1}$	$(2 \times 0.006)^{-1}$

Now, using (22) we have two pairs of (f_{GW}, t) values for two unknowns (t_0, M_c) . Expressing (22) for both $t_{\overline{AB}}$ and $t_{\overline{CD}}$ and dividing the two equations we obtain:

$$t_0 = \frac{A t_{\overline{CD}} - t_{\overline{AB}}}{A - 1}, \quad A \equiv \left(\frac{f_{\text{GW}}(t_{\overline{AB}})}{f_{\text{GW}}(t_{\overline{CD}})}\right)^{-8/3}. \quad (23)$$

Replacing by the numerical values, $A \simeq 2.95$ and $t_0 \simeq 0.054$ s. Now we can use (22) for either of the two values $t_{\overline{AB}}$ or $t_{\overline{CD}}$ and determine M_c . One obtains for the chirp mass

$$M_c \simeq 6 \times 10^{31} \text{ kg} \simeq 30 \times M_{\odot}. \quad (24)$$

Thus, the total mass M is

$$M = 4^{3/5} M_c \simeq 69 \times M_{\odot}. \quad (25)$$

This result is actually remarkably close to the best estimates using the full theory of General Relativity! [Even though the actual objects do not have precisely equal masses and the theory we have just used is not valid very close to the collision.]

B.5

$$M_c \simeq 30 \times M_\odot, \quad M \simeq 69 \times M_\odot.$$

1.0pt

B.6 From (8), Kepler's law states that $L = (GM/\Omega^2)^{1/3}$. The second pair of points highlighted in the plot correspond to the cycle prior to merger. Thus, we use (19) to obtain the orbital angular velocity at t_{CD} :

$$\Omega_{t_{\text{CD}}} \sim 2.6 \times 10^2 \text{ rad/s}. \quad (26)$$

Then, using the total mass (25) we find

$$L \sim 5 \times 10^2 \text{ km}. \quad (27)$$

Thus, these objects have a maximum radius of $R_{\text{max}} \sim 250 \text{ km}$. Hence they have over 30 times more mass and,

$$\frac{R_\odot}{R_{\text{max}}} \sim 3 \times 10^3 \quad (28)$$

they are 3000 times smaller than the Sun and!

Their linear velocity is

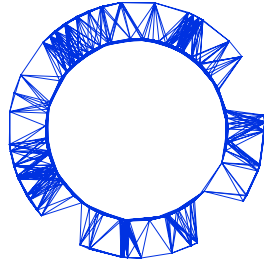
$$v_{\text{col}} = \frac{L}{2} \Omega \simeq 7 \times 10^4 \text{ km/s}. \quad (29)$$

They are moving at over 20% of the velocity of light!

B.6

$$L_{\text{collision}} \sim 5 \times 10^2 \text{ km}, \quad \frac{R_\odot}{R_{\text{max}}} \sim 3 \times 10^3, \quad \frac{v_{\text{col}}}{c} \sim 0.2.$$

1.0pt



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Detailed Marking Scheme Theory Problem 1

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GW150914 (10 points)

Part A. Newtonian (conservative) orbits (3.0 points)

A.1

Apply Newton's law to body 1	0.3
Relate the particle positions	0.3
Final expression (identification of n and α)	0.4
Total	1.0

[Give full points to final answer without derivation]

A.2

The total energy of the system	0.5
Obtain $M_1 r_1^2 + M_2 r_2^2 = \mu L^2$	0.2
Final expression for the energy (identification of $A(\mu, \Omega, L)$)	0.3
Total	1.0

[Give full points to final answer without derivation]

A.3

Equate the gravitational acceleration to the centripetal acceleration	0.5
Obtain E (identification of β)	0.5
Total	1.0

[Give full points to final answer without derivation; it is accepted to use virial theorem]

Part B - Introducing relativistic dissipation (7.0 points)

B.1

Cartesian components for \vec{r}_1 and \vec{r}_2	0.2
Expression for Q_{ij}	0.5
Simplified expression for Q_{ij} (identification of k, a_i, b_i, c_{ij})	0.3
Total	1.0

B.2

Third order derivative	0.5
Summation to obtain dE/dt (identification of ξ)	0.5
Total	1.0

B.3

Obtain dE/dt	0.2
Identify the power loss with the luminosity of GWs	0.2
Write expressions for L and dL/dt	0.2
Find the expression for $(d\Omega/dt)^3$	0.2
Expression for M_c	0.2
Total	1.0

B.4

Relate the frequency of the GWs with the orbital frequency	0.5
Write eq. (10) in question sheet as $d\Omega/dt = \chi\Omega^{11/3}$	0.5
Usage of (11) in the question sheet	0.5
Expression for the GW frequency (identification of p)	0.5
Total	2.0

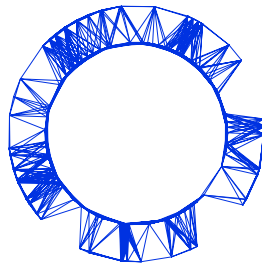
[dimensional analysis to determine p is OK]

B.5

Estimate frequencies and time averages	0.3
Obtain t_0 and A	0.3
Value for the chirp mass	0.3
Total mass	0.1
Total	1.0

B.6

Angular orbital velocity	0.3
Obtain L and R_{\max}	0.3
Value for the collision velocity	0.4
Total	1.0



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Solutions to Theory Problem 2

Where is the neutrino?

(Miguel C N Fiolhais and António Onofre)

July 24, 2018

v1.2

Where is the neutrino? (10 points)

Part A. ATLAS Detector physics (4.0 points)

A.1

The magnetic force is the centripetal force:

$$m \frac{v^2}{r} = evB \Rightarrow r = \frac{mv}{eB}.$$

First express the velocity in terms of the kinetic energy,

$$K = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2K}{m}},$$

and then insert it in the expression above for the radius to get

A.1

$$r = \frac{\sqrt{2Km}}{eB}.$$

0.5pt

A.2

The radius of the circular motion of a charged particle in the presence of a uniform magnetic field is given by,

$$r = \frac{mv}{eB}.$$

This formula is valid in the relativistic scenario if the mass correction, $m \rightarrow \gamma m$ is included:

$$r = \frac{\gamma mv}{eB} = \frac{p}{eB} \Rightarrow p = reB.$$

Note that the radius of the circular motion is half the radius of the inner part of the detector. One obtains [1 MeV/c = 5.34 × 10⁻²² m kg s⁻¹]

A.2

$$p = 330 \text{ MeV}/c.$$

0.5pt

A.3

The acceleration for the particle is $a = \frac{evB}{\gamma m} \sim \frac{ecB}{\gamma m}$, in the ultrarelativistic limit. Then,

$$P = \frac{e^4 c^2 \gamma^4 B^2}{6\pi\epsilon_0 c^3 \gamma^2 m^2} = \frac{e^4 \gamma^2 c^4 B^2}{6\pi\epsilon_0 c^5 m^2}.$$

Since $E = \gamma mc^2$ we can obtain $\gamma^2 c^4 = \frac{E^2}{m^2}$ and, finally,

$$P = \frac{e^4}{6\pi\epsilon_0 m^4 c^5} E^2 B^2.$$

Therefore,

A.3

$$\xi = \frac{1}{6\pi}, \quad n = 5 \quad \text{and} \quad k = 4.$$

1.0pt

A.4

The power emitted by the particle is given by,

$$P = -\frac{dE}{dt} = \frac{e^4}{6\pi\epsilon_0 m^4 c^5} E^2 B^2.$$

The energy of the particle as a function of time can be calculated from

$$\int_{E_0}^{E(t)} \frac{1}{E^2} dE = -\int_0^t \frac{e^4}{6\pi\epsilon_0 m^4 c^5} B^2 dt,$$

where $E(0) = E_0$. This leads to,

$$\frac{1}{E(t)} - \frac{1}{E_0} = \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5} t \quad \Rightarrow \quad E(t) = \frac{E_0}{1 + \alpha E_0 t},$$

with

A.4

$$\alpha = \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5}.$$

1.0pt

A.5

If the initial energy of the electron is 100 GeV, the radius of curvature is extremely large ($r = \frac{E}{eBc} \approx 167$ m). Therefore, in approximation, one can consider the electron is moving in the inner part of the ATLAS detector along a straight line. The time of flight of the electron is $t = R/c$, where $R = 1.1$ m is the radius of the inner part of the detector. The total energy lost due to synchrotron radiation is,

$$\Delta E = E(R/c) - E_0 = \frac{E_0}{1 + \alpha E_0 \frac{R}{c}} - E_0 \approx -\alpha E_0^2 \frac{R}{c}$$

and

A.5

$$\Delta E = -56 \text{ MeV}.$$

0.5pt

A.6

In the ultrarelativistic limit, $v \approx c$ and $E \approx pc$. The cyclotron frequency is,

$$\omega(t) = \frac{c}{r(t)} = \frac{ecB}{p(t)} = \frac{ec^2B}{E(t)}$$

A.6

$$\omega(t) = \frac{ec^2B}{E_0} \left(1 + \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5} E_0 t \right).$$

0.5pt

Part B. Finding the neutrino (6.0 points)

B.1

Since the W^+ boson decays into an anti-muon and a neutrino, one can use principles of conservation of energy and linear momentum to calculate the unknown $p_z^{(\nu)}$ of the neutrino. Moreover, the anti-muon and the neutrino can be considered massless, which implies that the magnitude of their momenta (times c) and their energies are the same. Therefore, the conservation of linear momentum can be expressed as

$$\vec{p}^{(W)} = \vec{p}^{(\mu)} + \vec{p}^{(\nu)},$$

and the conservation of energy as,

$$E^{(W)} = cp^{(\mu)} + cp^{(\nu)}.$$

In addition, one can also relate the energy and the momentum of the W^+ boson through its mass,

$$m_W^2 = (E^{(W)})^2/c^4 - (p^{(W)})^2/c^2$$

which leads to a quadratic equation on $p_z^{(\nu)}$,

$$\begin{aligned} m_W^2 &= [(p^{(\mu)} + p^{(\nu)})^2 - (\vec{p}^{(\mu)} + \vec{p}^{(\nu)})^2] / c^2 \\ &= (2p^{(\mu)}p^{(\nu)} - 2\vec{p}^{(\mu)} \cdot \vec{p}^{(\nu)}) / c^2 \end{aligned}$$

B.1

$$m_W^2 = \frac{1}{c^2} \left(2p^{(\mu)} \sqrt{(p_T^{(\nu)})^2 + (p_z^{(\nu)})^2} - 2\vec{p}_T^{(\mu)} \cdot \vec{p}_T^{(\nu)} - 2p_z^{(\mu)} p_z^{(\nu)} \right).$$

1.5pt

B.2

The numerical substitution directly in the answer of B.1, using

$$p^{(\mu)} = 37.2 \text{ GeV}/c \quad m_W^2 c^2 = 6464.2 (\text{GeV}/c)^2 \quad p_T^{(\nu)2} = 10864.9 (\text{GeV}/c)^2$$

$$\vec{p}_T^{(\mu)} \cdot \vec{p}_T^{(\nu)} = 2439.3 (\text{GeV}/c)^2 \quad p_z^{(\mu)} = -12.4 \text{ GeV}/c,$$

leads to

$$6464.2 = 74.4 \sqrt{10864.9 + p_z^{(\nu)2}} - 4878.6 + 24.8 p_z^{(\nu)}.$$

This is a quadratic equation, equivalent to

$$0.88889 p_z^{(\nu)2} + 101.64 p_z^{(\nu)} - 12378 = 0$$

whose solutions are:

B.2

1.5pt

$$p_z^{(\nu)} = 74.0 \text{ GeV}/c \quad \text{or} \quad p_z^{(\nu)} = -188.3 \text{ GeV}/c.$$

The general solution of the equation above in B.1 leads to

$$p_z^{(\nu)} = \frac{2\vec{p}_T^{(\mu)} \cdot \vec{p}_T^{(\nu)} p_z^{(\mu)} + m_W^2 c^2 p_z^{(\mu)}}{2(p_T^{(\mu)})^2} \pm \frac{p^{(\mu)} \sqrt{-4(p_T^{(\mu)})^2 (p_T^{(\nu)})^2 + 4(\vec{p}_T^{(\mu)} \cdot \vec{p}_T^{(\nu)})^2 + 4\vec{p}_T^{(\mu)} \cdot \vec{p}_T^{(\nu)} m_W^2 c^2 + m_W^4 c^4}}{2(p_T^{(\mu)})^2}$$

Numerical substitution leads to the above mentioned values for $p_z^{(\nu)}$.

B.3

The final state particles of the top quark decay are the anti-muon, the neutrino and jet 1. Since the neutrino is now fully reconstructed the energy and linear momentum of the top quark can be calculated as,

$$\begin{aligned} E^{(t)} &= cp^{(\mu)} + cp^{(\nu)} + cp^{(j_1)} \\ \vec{p}^{(t)} &= \vec{p}^{(\mu)} + \vec{p}^{(\nu)} + \vec{p}^{(j_1)}. \end{aligned}$$

The top quark mass is,

$$\begin{aligned} m_t &= \sqrt{(E^{(t)})^2/c^4 - (\vec{p}^{(t)})^2/c^2} \\ &= c^{-1} \sqrt{(p^{(\mu)} + p^{(\nu)} + p^{(j_1)})^2 - (\vec{p}^{(\mu)} + \vec{p}^{(\nu)} + \vec{p}^{(j_1)})^2}. \end{aligned}$$

The substitution of values leads to two possible masses:

B.3

1.0pt

$$m_t = 169.3 \text{ GeV}/c^2 \quad \text{or} \quad m_t = 311.2 \text{ GeV}/c^2$$

B.4

According to the frequency distribution for signal (dashed line), the probability of the $m_t = 169.3 \text{ GeV}/c^2$ solution is roughly 0.1 while the probability of the $m_t = 311.2 \text{ GeV}/c^2$ solution is below 0.01. Therefore,

B.4

The most likely candidate is the $m_t = 169.3 \text{ GeV}/c^2$ solution.

1.0pt

B.5

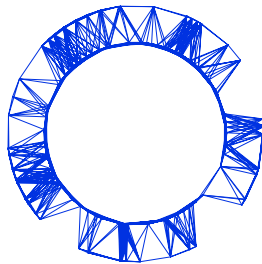
The top quark energy for the most likely candidate is $E^{(t)} = cp^{(\mu)} + cp^{(\nu)} + cp^{(j_1)} = 272.6 \text{ GeV}$.

$$d = vt = v\gamma t_0 = \frac{p^{(t)}}{m_t} t_0 = ct_0 \sqrt{\frac{E^{(t)^2}}{m_t^2 c^4} - 1}.$$

B.5

$$d = 2 \times 10^{-16} \text{ m}.$$

1.0pt



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Detailed Marking Scheme Theory Problem 2

Where is the neutrino?

(Miguel C N Fiolhais and António Onofre)

July 24, 2018

v1.2

Where is the neutrino? (10 points)

Part A. ATLAS Detector physics (4.0 points)

A.1

Magnetic force as centripetal force	0.2
Velocity in terms of the kinetic energy	0.2
Final expression for the radius	0.1
Total	0.5

A.2

Recognize that $m \rightarrow \gamma m$	0.2
Expression for p	0.1
Radius of the circular motion is half the radius of the inner part of the detector	0.1
Final numerical expression for p	0.1
Total	0.5

A.3

Acceleration for the particle	0.2
Replace a in the given expression for P	0.2
Use $E = \gamma mc^2$	0.2
Final expression for P with identification of ξ, n, k	0.4
Total	1.0

A.4

Power emitted by the particle	0.3
Solve the integral to get $\frac{1}{E(t)} - \frac{1}{E_0} = \alpha t$	0.4
Arrive at expression for $E(t)$ and identify α	0.3
Total	1.0

A.5

Electron moving in a straight line	0.1
Time of flight	0.1
Expression for the energy lost	0.2
Final numerical value for energy lost	0.1
Total	0.5

A.6

$v \simeq c$ (ultrarelativistic limit)	0.1
$E \simeq pc$ (ultrarelativistic limit)	0.1
Expression for the frequency	0.3
Total	0.5

Part B. Finding the neutrino (6.0 points)

B.1

Expression for $\vec{p}^{(W)}$	0.3
Expression for $E^{(W)}$	0.3
Write the initial expression for m_W^2	0.3
Arrive at the final expression for m_W^2	0.6
Total	1.5

B.2

Numerical values for $p^{(\mu)}, m_W^2 c^2, p_T^{(\nu)2}, \vec{p}_T^{(\mu)} \cdot \vec{p}_T^{(\nu)}, p_z^{(\mu)}$ (0.2 points each)	1.0
Get the two numerical solutions of the quadratic equation	0.5
Total	1.5

B.3

Expression for $E^{(t)}$	0.1
Expression for $\vec{p}^{(t)}$	0.1
Write the initial expression for m_t	0.2
Arrive at the two possible masses	0.6
Total	1.0

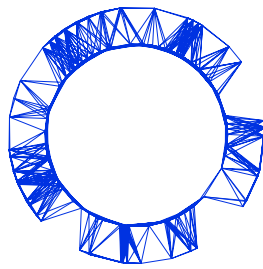
B.4

Estimate the probability for the lighter mass	0.3
Estimate the probability for the heavier mass	0.3
Conclude about the most likely candidate	0.4
Total	1.0

B.5

Numerical value for $E^{(t)}$	0.4
Analytical expression for d	0.4
Numerical value for d	0.2
Total	1.0

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Solutions to Theory Problem 3

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Physics of Live Systems

(Rui Travasso, Lucília Brito)

July 24, 2018

v1.0

Physics of Live Systems (10 points)

Part A. The physics of blood flow (4.5 points)

A.1

Since the vessel network is symmetrical, the flow in a vessel of level $i + 1$ is half the flow in a vessel of level i .

In this way, we can sum the pressure differences in all levels:

$$\Delta P = \sum_{i=0}^{N-1} Q_i R_i = Q_0 \sum_{i=0}^{N-1} \frac{R_i}{2^i}.$$

Introducing the radii dependences yields

$$\Delta P = Q_0 \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = Q_0 \frac{8\ell_0 \eta}{\pi r_0^4} \sum_{i=0}^{N-1} \frac{2^{4i/3}}{2^i 2^{i/3}} = Q_0 N \frac{8\ell_0 \eta}{\pi r_0^4}.$$

Therefore

$$Q_0 = \Delta P \frac{\pi r_0^4}{8N\ell_0 \eta}.$$

Hence, the flow rate for a vessel network in level i is

A.1

1.3pt

$$Q_i = \Delta P \frac{\pi r_0^4}{2^{i+3} N \ell_0 \eta}.$$

A.2

Replace values in the formula and change units appropriately

$$\begin{aligned} Q_0 &= \frac{\Delta P \pi r_0^4}{8N\ell_0 \eta} = \\ &= \frac{(55 - 30) \times 1.013 \times 10^5 \times 3.1415 \times (6.0 \times 10^{-5})^4}{760 \times 48 \times 2.0 \times 10^{-3} \times 3.5 \times 10^{-3}} = 4.0 \times 10^{-10} \text{ m}^3/\text{s} \end{aligned}$$

to obtain the final value in the requested unites:

A.2

0.5pt

$$Q_0 \simeq 1.5 \text{ ml/h}.$$

A.3

The current is given by

$$I = \frac{P_{\text{in}} e^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}}.$$

The pressure difference in the capacitor is

$$P_{\text{out}} e^{i(\omega t + \phi)} = \frac{P_{\text{in}} e^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} \frac{1}{i\omega C} = \frac{P_{\text{in}} e^{i\omega t}}{i\omega C R - \omega^2 LC + 1}.$$

The amplitude is

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}}.$$

To be smaller than P_{in} , for $\omega \rightarrow 0$:

$$(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2 > 1 \iff -2CL + C^2 R^2 > 0.$$

Replacing the expressions for L , C , and R we get: $\frac{64\eta^2 \ell^2}{3Ehr^3\rho} > 1$.

A.3
2.0pt

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}}.$$

Condition:

$$\frac{64\eta^2 \ell^2}{3Ehr^3\rho} > 1.$$

Alternative way to obtain P_{out} :

The amplitude of the current in the equivalent circuit is $I_0 = \frac{P_{\text{in}}}{Z}$, where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

is the modulus of the impedance. Hence, the voltage amplitude in the capacitor is

$$P_{\text{out}} = \frac{1}{\omega C} \times I_0 = \frac{P_{\text{in}}}{\sqrt{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}}.$$

A.4

The previous condition can also be expressed as

$$h < \frac{64\eta^2 \ell^2}{3Er^3\rho}.$$

For the network referred to in **A.2**

$$h < \frac{64\eta^2 \ell_0^2 \times 2^i}{3 \times 2^{2i/3} E r_0^3 \rho} = \frac{64 \times (3.5 \times 10^{-3})^2 \times (2.0 \times 10^{-3})^2}{3 \times 0.06 \times 10^6 \times (6.0 \times 10^{-5})^3 \times 1.05 \times 10^3} \times 2^{i/3} = 7.7 \times 10^{-5} \times 2^{i/3}.$$

For $i = 0$, in the worse case scenario,

$$h_{\max} = 7.7 \times 10^{-5} \times 2^0 = 7.7 \times 10^{-5} \text{ m}$$

This value is certainly observed in these vessels since their radius range from $18 \mu\text{m}$ to $60 \mu\text{m}$. A wall width smaller than $80 \mu\text{m}$ is certainly reasonable.

A.4 Maximum $h = 8 \times 10^{-5} \text{ m}$

0.7pt

Part B. Tumor growth (5.5 points)

B.1

The expressions for the masses of tumour and normal tissue are written as:

$$\begin{cases} M_T = V_T \rho_T = V_T \rho_0 \left(1 + \frac{p}{K_T}\right) \\ M_N = V \rho_0 = (V - V_T) \rho_0 \left(1 + \frac{p}{K_N}\right) \end{cases}$$

The pressure, p , can be expressed as

$$p = \frac{M_T K_T}{V_T \rho_0} - K_T$$

and, then, used in the equation for M_N :

$$M_N = (V - V_T) \frac{M_N}{V} \left[\left(1 - \frac{K_T}{K_N}\right) + \frac{M_T V K_T}{V_T M_N K_N} \right]$$

Simplifying and rearranging the terms, the equation for v becomes

$$(1 - \kappa) v^2 - (1 + \mu) v + \mu = 0,$$

for which the solution is (the other solution of the quadratic equation is not physically relevant since does not lead to $v = 0$ for $\mu = 0$)

B.1

$$v = \frac{1 + \mu - \sqrt{(1 + \mu)^2 - 4\mu(1 - \kappa)}}{2(1 - \kappa)}.$$

1.0pt

B.2

For $r < R_T$, the conservation of energy implies that

$$4\pi r^2 (-k) \frac{dT}{dr} = \mathcal{P} \frac{4}{3} \pi r^3.$$

Therefore, the temperature difference to $37\text{ }^\circ\text{C} = 310.15\text{ K}$, $\Delta T(r)$, is given by

$$\Delta T(r) = -\frac{\mathcal{P}r^2}{6k} + C,$$

where C is a constant.

For $r > R_T$, the conservation of energy implies that

$$4\pi r^2(-k)\frac{dT}{dr} = \mathcal{P}\frac{4}{3}\pi R_T^3.$$

Therefore, the temperature difference to $37\text{ }^\circ\text{C}$ is

$$\Delta T(r) = \frac{\mathcal{P}R_T^3}{3kr}.$$

In this case there is no constant, since very far away the increase in temperature is zero.

Matching the two solutions at $r = R_T$ gives

$$C = \frac{\mathcal{P}R_T^2}{2k}.$$

Therefore the temperature at the centre of the tumour, in SI units, is

B.2	Temperature: $310.15 + \frac{\mathcal{P}R_T^2}{2k}$.	1.7pt
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B.3

The increase in temperature at the tumour surface (the lower temperature in the tumour) is

$$\Delta T(R_T) = \frac{\mathcal{P}R_T^2}{3k}.$$

This increase should be equal to 6.0 K . Therefore,

$$\mathcal{P} = \frac{3\Delta Tk}{R_T^2} = \frac{3 \times 6 \times 0.6}{0.05^2} = 4.3\text{ kW/m}^3.$$

B.3	$\mathcal{P}_{\min} = 4.3\text{ kW/m}^3$.	0.5pt
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B.4

We can relate δr with the pressure in the tumour, using the relation given in the text up to leading order in $p - P_{\text{cap}}$: $\delta r = \frac{p - P_{\text{cap}}}{2(p_c - P_{\text{cap}})} \delta r_c$. Therefore, if $p - P_{\text{cap}}$ is very small, also it is δr .

The pressure can be related with the volume. We know that

$$\frac{M_N}{V_N} = \frac{\rho_0 V}{V - V_T} = \frac{\rho_0}{1 - v} = \rho_0 \left(1 + \frac{p}{K_N}\right).$$

And so $p = \frac{K_N v}{1-v}$.

When the thinner vessels are narrower, the flow rate in the main vessel is altered:

$$\Delta P = (Q_0 + \delta Q_0) \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = (Q_0 + \delta Q_0) \frac{8\ell_0 \eta}{\pi r_0^4} \left(\sum_{i=0}^{N-2} \frac{2^{4i/3}}{2^i 2^{i/3}} + \frac{2^{4(N-1)/3}}{2^{N-1} 2^{(N-1)/3} \left(1 - \frac{\delta r}{r_0/2^{(N-1)/3}}\right)^4} \right)$$

$$\Rightarrow \Delta P \simeq (Q_0 + \delta Q_0) \frac{\Delta P}{NQ_0} \left(N - 1 + 1 + \frac{4\delta r}{r_{N-1}} \right)$$

Noting that $\frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{\delta Q_0}{Q_0}$, we obtain

$$1 + \frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{1}{1 + \frac{4\delta r}{Nr_{N-1}}} \simeq 1 - \frac{4\delta r}{Nr_{N-1}}.$$

And so:

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{4}{N} \frac{\delta r}{r_{N-1}}.$$

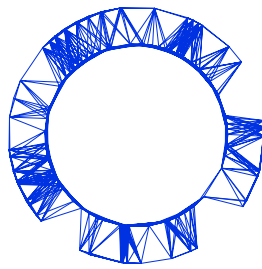
Putting all together

B.4

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{2}{N} \frac{K_N v - (1-v)P_{\text{cap}}}{(1-v)(p_c - P_{\text{cap}})} \frac{\delta r_c}{r_{N-1}}.$$

2.3pt

**IPhO 2018
Lisbon, Portugal**



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Detailed Marking Scheme Theory Problem 3

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Physics of Live Systems

(Rui Travasso, Lucília Brito)

July 13, 2018

v1.0

Physics of Live Systems (10 points)

Part A. The physics of blood flow (4.5 points)

A.1

Flow in a $i + 1$ level vessel is half of flow in a i level vessel	0.3
Obtain ΔP for all levels	0.3
Introduce radii dependences to obtain ΔP	0.3
Expression for Q_0	0.2
Expression for Q_i	0.2
Total	1.3

A.2

Replace values in formula with the correct units	0.3
Obtain final value in the requested units	0.2
Total	0.5

A.3

Obtain the current	0.5
Final expression for P_{out}	0.5
Arrive at intermediate condition $-2CL + C^2R^2 > 0$	0.5
Obtain the condition $\frac{64\eta^2\ell^2}{3Ehr^3\rho} > 1$	0.5
Total	2.0

A.4

Solve inequation in A.3 in order to h	0.2
Replace the values for the network	0.3
Obtain the maximum value for h	0.2
Total	0.7

Part B. Tumor growth (5.5 points)

B.1

Expressions for the masses of tumour and normal tissues (0.1 each)	0.2
Solve for the pressure	0.2
Equation without pressure	0.2
Solve to obtain the final solution for v (if not in reduced variables discount 0.2)	0.4
Total	1.0

B.2

Conservation of energy for $r < R_T$	0.4
Solve to obtain the temperature difference to 37 °C	0.2
Conservation of energy for $r > R_T$	0.4
Solve to obtain the temperature difference to 37 °C	0.2
Find the integration constant, C	0.2
Final result for $T(r = 0)$	0.3
Total	1.7

[Forget to add body temperature (between this question and the question below), discount 0.5 pt.]

B.3

Consider the increase of temperature at the tumour surface	0.2
Equate the temperature increase to 6.0 K	0.1
Obtain the numerical value for the power	0.2
Total	0.5

[Forget to add body temperature (between this question and the question above), discount 0.5 pt.]

B.4

Relate δr with pressure in tumour, up to leading order in $p - P_{\text{cap}}$	0.3
Relate pressure with tumour volume (discount 0.2 if p is not written as function of reduced variables)	0.4
Use the result of part A to calculate the new flow (include correctly the radius change at the capillary level)	0.3
Notice that $\frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{\delta Q_0}{Q_0}$	0.3
Obtain $\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{4}{N} \frac{\delta r}{r_{N-1}}$	0.6
Final result for $\delta Q_{N-1}/Q_{N-1}$	0.4
Total	2.3