Dark Matter

A. Cluster of Galaxies

Question A.1

Answer	Marks
Potential energy for a system of a spherical object with mass $M(r) = \frac{4}{3}\pi r^3 \rho$ and a test particle with mass *dm* at a distance *r* is given by $$dU = -G\frac{M(r)}{r}dm$$	0.2 pts
Thus for a sphere of radius *R* $$U = -\int_0^R G\frac{M(r)}{r}dm = -\int_0^R G\frac{4\pi r^3\rho}{3r}4\pi r^2\rho dr = -\frac{16}{3}G\pi^2\rho^2\int_0^R r^4 dr$$ $$= -\frac{16}{15}G\pi^2\rho^2 R^5$$	0.6 pts
Then using the total mass of the system $$M = \frac{4}{3}\pi R^3\rho$$ we have $$U = -\frac{3}{5}\frac{GM^2}{R}$$	0.2 pts
Total	1.0 pts

Question A.2

<table>
<tr><th>Answer</th><th>Marks</th></tr>
<tr><td>Using the Doppler Effect,
$$f_i = f_0 \frac{1}{1+\beta} \approx f_0(1-\beta),$$
where $\beta = v/c$ and $v << c$. Thus the *i*-th galaxy moving away (radial) speed is
$$V_{ri} = -\frac{f_i - f_0}{f_0} c$$
Alternative without approximation:
$$f_i = f_0 \frac{1}{1+\beta}$$
$$V_{ri} = c\left(\frac{f_0}{f_i} - 1\right)$$</td><td>0.2 pts</td></tr>
<tr><td>All the galaxies in the galaxy cluster will be moving away together due to the cosmological expansion. Thus the average moving away speed of the N galaxies in the cluster is
$$V_{cr} = -\frac{c}{Nf_0}\sum_{i=1}^{N}(f_i - f_0) = -\frac{c}{N}\sum_{i=1}^{N}\left(\frac{f_i}{f_0} - 1\right).$$
Alternative without approximation:
$$V_{cr} = \frac{cf_0}{N}\sum_{i=1}^{N}\left(\frac{1}{f_i} - \frac{1}{f_0}\right) = \frac{c}{N}\sum_{i=1}^{N}\left(\frac{f_0}{f_i} - 1\right)$$</td><td>0.3 pts</td></tr>
<tr><td>Total</td><td>0.5 pts</td></tr>
</table>

Question A.3

Answer	Marks
The galaxy moving away speed V_i, in part A.2, is only one component of the three component of the galaxy velocity. Thus the average square speed of each galaxy with respect to the center of the cluster is $\frac{1}{N}\sum_{i=1}^{N}(\vec{V}_i-\vec{V}_c)^2=\frac{1}{N}\sum_{i=1}^{N}(V_{xi}-V_{xc})^2+(V_{yi}-V_{yc})^2+(V_{zi}-V_{zc})^2$ Due to isotropic assumption $\frac{1}{N}\sum_{i=1}^{N}(\vec{V}_i-\vec{V}_c)^2=\frac{3}{N}\sum_{i=1}^{N}(V_{ri}-V_{cr})^2$	0.5 pts
And thus the root mean square of the galaxy speed with respect to the cluster center is $v_{rms}=\sqrt{\frac{3}{N}\sum_{i=1}^{N}(V_{ri}-V_{rc})^2}=\sqrt{\frac{3}{N}\sum_{i=1}^{N}(V_{ri}^{\;2}-2V_{cr}V_{ri}+V_{cr}^{\;2})}=\sqrt{\frac{3}{N}\left(\sum_{i=1}^{N}V_{ri}^{\;2}\right)-3V_{cr}^{\;2}}$ $v_{rms}=c\sqrt{3}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_i}{f_0}-1\right)^2\right)-\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_i}{f_0}-1\right)\right)^2}$ $=\frac{c\sqrt{3}}{f_0}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(f_i^2-2f_if_0+f_0^{\;2}\right)\right)-\left(\left(\frac{1}{N}\sum_{i=1}^{N}f_i\right)^2-2\frac{f_0}{N}\sum_{i=1}^{N}f_i+f_0^{\;2}\right)}$ $=\frac{c\sqrt{3}}{f_0N}\sqrt{\left(N\sum_{i=1}^{N}f_i^2\right)-\left(\sum_{i=1}^{N}f_i\right)^2}$ Alternative without approximation:	0.7 pts

$$v_{rms} = c\sqrt{3}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_0}{f_i}-1\right)^2\right)-\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_0}{f_i}-1\right)\right)^2}$$ $$=\frac{c\sqrt{3}}{f_0}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{f_i^2}-2\frac{1}{f_i}\frac{1}{f_0}+\frac{1}{f_0^2}\right)\right)-\left(\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{f_i}\right)^2-2\frac{1}{N}\frac{1}{f_0}\sum_{i=1}^{N}\frac{1}{f_i}+\frac{1}{f_0^2}\right)}$$ $$=\frac{cf_0\sqrt{3}}{N}\sqrt{\left(N\sum_{i=1}^{N}\left(\frac{1}{f_i}\right)^2\right)-\left(\sum_{i=1}^{N}\frac{1}{f_i}\right)^2}$$	
The mean kinetic energy of the galaxies with respect to the center of the cluster is $$K_{ave} = \frac{m}{2}\frac{1}{N}\sum_{i=1}^{N}(\vec{V}_i-\vec{V}_c)^2 = \frac{m}{2}v_{rms}^{\ 2}$$	0.3 pts
Total	1.5 pts

Question A.4

Answer	Marks
The time average of $d\Gamma/dt$ vanishes $$\left\langle \frac{d\Gamma}{dt} \right\rangle_t = 0$$ Now $$\frac{d\Gamma}{dt} = \frac{d}{dt}\sum_i \vec{p}_i \cdot \vec{r}_i = \sum_i \frac{d\vec{p}_i}{dt}\cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \frac{d\vec{r}_i}{dt}$$ $$= \sum_i \vec{F}_i \cdot \vec{r}_i + \sum_i m_i \vec{v}_i \cdot \vec{v}_i = \sum_i \vec{F}_i \cdot \vec{r}_i + 2K$$	0.6 pts
Where *K* is the total kinetic energy of the system. Since the gravitational force on *i*-th particle comes from its interaction with other particles then $$\sum_i \vec{F}_i \cdot \vec{r}_i = \sum_{i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_i = \sum_{i<j} \vec{F}_{ji} \cdot \vec{r}_i - \sum_{i>j} \vec{F}_{ij} \cdot \vec{r}_i = \sum_{i<j} \vec{F}_{ji} \cdot \vec{r}_i - \sum_{i<j} \vec{F}_{ji} \cdot \vec{r}_j$$ $$= \sum_{i<j} \vec{F}_{ji} \cdot (\vec{r}_i - \vec{r}_j) = -\sum_{i<j} G \frac{m_i m_j}{\lvert \vec{r}_i - \vec{r}_j \rvert^2} \frac{(\vec{r}_i - \vec{r}_j)}{\lvert \vec{r}_i - \vec{r}_j \rvert} \cdot (\vec{r}_i - \vec{r}_j) = -\sum_{i<j} G \frac{m_i m_j}{\lvert \vec{r}_i - \vec{r}_j \rvert} = U_{\text{tot}}$$ Alternative proof: $$\sum_i \vec{F}_i . \vec{r}_i = \sum_{i,j\neq i} \vec{F}_{ji} . \vec{r}_i = \vec{F}_{21}.\vec{r}_1 + \vec{F}_{31}.\vec{r}_1 + \vec{F}_{41}.\vec{r}_1 + \cdots + \vec{F}_{N1}.\vec{r}_1 +$$ $$\vec{F}_{12}.\vec{r}_2 + \vec{F}_{32}.\vec{r}_2 + \vec{F}_{42}.\vec{r}_2 + \cdots + \vec{F}_{N2}.\vec{r}_2 +$$ $$\vec{F}_{13}.\vec{r}_3 + \vec{F}_{23}.\vec{r}_3 + \vec{F}_{43}.\vec{r}_3 + \cdots + \vec{F}_{N3}.\vec{r}_3 + \ldots$$ $$\vec{F}_{1N}.\vec{r}_N + \vec{F}_{2N}.N_N + \vec{F}_{3N}.\vec{r}_N + \cdots + \vec{F}_{NN-1}.\vec{r}_{N-1}$$ Collecting terms and noting that $\vec{F}_{ij} = -\vec{F}_{ji}$ we have	0.9 pts

$\vec{F}_{12}.(\vec{r}_2-\vec{r}_1)+\vec{F}_{13}.(\vec{r}_3-\vec{r}_1)+\vec{F}_{14}.(\vec{r}_4-\vec{r}_1)+\cdots+\vec{F}_{23}.(\vec{r}_3-\vec{r}_2)$ $+\vec{F}_{24}.(\vec{r}_4-\vec{r}_2)+\cdots+\vec{F}_{34}.(\vec{r}_4-\vec{r}_3)+\cdots=\sum_{i<j}\vec{F}_{ji}.(\vec{r}_i-\vec{r}_j)$ $=-\sum_{i<j}G\frac{m_im_j}{\lvert\vec{r}_i-\vec{r}_j\rvert^2}\frac{(\vec{r}_i-\vec{r}_j)}{\lvert\vec{r}_i-\vec{r}_j\rvert}.(\vec{r}_i-\vec{r}_j)=-\sum_{i<j}G\frac{m_im_j}{\lvert\vec{r}_i-\vec{r}_j\rvert}=U_{tot}$	
Thus we have $$\frac{d\Gamma}{dt}=U+2K$$ And by taking its time average we obtain $\left\langle\frac{d\Gamma}{dt}=U+2K\right\rangle_t=0$ and thus $\langle K\rangle_t=-\frac{1}{2}\langle U\rangle_t$. Therefore $\gamma=\frac{1}{2}$.	0.2 pts
Total	1.7 pts

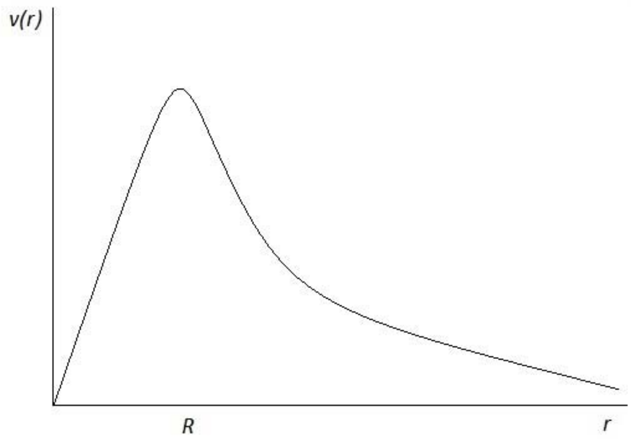
Question A.5

Answer	Marks
Using Virial theorem, and since the dark matter has the same root mean square speed as the galaxy, then we have $$\langle K \rangle_t = -\frac{1}{2}\langle U \rangle_t$$ $$\frac{M}{2}v_{rms}^2 = \frac{1}{2}\frac{3}{5}\frac{GM^2}{R}$$	0.3 pts
From which we have $$M = \frac{5Rv_{rms}^2}{3G}$$	0.1 pts
And the dark matter mass is then $$M_{dm} = \frac{5Rv_{rms}^2}{3G} - Nm_g$$	0.1 pts
Total	0.5 pts

B. Dark Matter in a Galaxy

Question B.1

Answer	Marks
Answer B.1: The gravitational attraction for a particle at a distance *r* from the center of the sphere comes only from particles inside a spherical volume of radius *r*. For particle inside the sphere with mass m_s , assuming the particle is orbiting the center of mass in a circular orbit, we have $$G\frac{m'(r)m_s}{r^2}=\frac{m_s v_0^2}{r}$$	0.3 pts
with $m'(r)$ is the total mass inside a sphere of radius r $$m'(r)=\frac{4}{3}\pi r^3 m_s n$$ Thus we have $$v(r)=\left(\frac{4\pi G n m_s}{3}\right)^{1/2} r$$	0.2 pts
While for particle outside the sphere, we have $$v(r)=\left(\frac{4\pi G n m_s R^3}{3r}\right)^{1/2}$$	0.2 pts

The sketch is given below  Sketch of the rotation velocity vs distance from the center of galaxy	0.1 pts
Total	0.8 pts

Question B.2

Answer	Marks
The total mass can be inferred from $$G\frac{m'(R_g)m_s}{R_g^{\ 2}}=\frac{m_s v_0^2}{R_g}$$ Thus $$m_R=m'(R_g)=\frac{v_0^2 R_g}{G}$$	0.5 pts
Total	0.5 pts

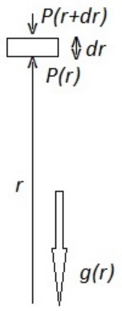
Question B.3

Answer	Marks
Base on the previous answer in B.1, if the mass of the galaxy comes only from the visible stars, then the galaxy rotation curve should fall proportional to $1/\sqrt{r}$ on the outside at a distance $r > R_g$. But in the figure of problem b) the curve remain constant after $r > R_g$, we can infer from $$G\frac{m'(r)m_s}{r^2} = \frac{m_s v_0^2}{r}.$$ to make $v(r)$ constant, then $m'(r)$ should be proportional to r for $r > R_g$, i.e. for $r > R_g$, $m'(r) = Ar$ with A is a constant.	0.3 pts
While for $r < R_g$, to obtain a linear plot proportional to r, then $m'(r)$ should be proportional to r^3, i.e. $m'(r) = Br^3$.	0.3 pts
Thus for $r < R_g$ we have $$m'(r) = \int_0^r \rho_t(r) 4\pi r'^2\, dr' = Br^3$$ $$dm'(r) = \rho_t(r) 4\pi r^2 dr = 3Br^2 dr$$ Thus total mass density $\rho_t(r) = \frac{3B}{4\pi}$	0.2 pts
$m_R = \int_0^{R_g} \frac{3B}{4\pi} 4\pi r'^2\, dr' = BR_g{}^3$ or $B = \frac{m_R}{R_g{}^3} = \frac{v_0^2}{GR_g{}^2}$ Thus the dark matter mass density $\rho(r) = \frac{3v_0^2}{4\pi G R_g{}^2} - nm_s$	0.2 pts

While for $r > R_g$ we have $m'(r) = \int_0^{R_g} \rho(r')4\pi r'^2\,dr' + \int_{R_g}^{r} \rho(r')4\pi r'^2\,dr' = Ar$ $m'(r) = m_R + \int_{R_g}^{r} \rho(r')4\pi r'^2\,dr' = Ar$ $\int_R^r \rho(r')4\pi r'^2\,dr' = Ar - M_0$ $\rho(r)4\pi r^2 = A$, or $\rho(r) = \frac{A}{4\pi r^2}$.	0.2 pts
Now to find the constant *A*. $\int_R^r \frac{A}{4\pi r'^2} 4\pi r'^2\,dr' = A(r - R_g) = Ar - m_R$ Thus $AR_g = m_R$ and $A = \frac{v_0^2}{G}$ We can also find *A* from the following $G\frac{m'(r)m_s}{r^2} = G\frac{Arm_s}{r^2} = \frac{m_s v_0^2}{r}$, thus $A = \frac{v_0^2}{G}$. Thus the dark matter mass density (which is also the total mass density since $n \approx 0$ for $r \geq R_g$. $\rho(r) = \frac{v_0^2}{4\pi G r^2}$ for $r \geq R_g$	0.3 pts
Total	1.5 pts

C. Interstellar Gas and Dark Matter

Question C.1

Answer	Marks
Consider a very small volume of a disk with area A and thickness Δr, see Fig.1  Figure 1. Hydrostatic equilibrium In hydrostatic equilibrium we have $$(P(r)-P(r+\Delta r))A-\rho g(r)A\Delta r=0$$	0.3 pts
$$\frac{\Delta P}{\Delta r}=-\rho\frac{Gm'(r)}{r^2}$$ $$\frac{dP}{dr}=-\rho\frac{Gm'(r)}{r^2}=-n(r)m_p\frac{Gm'(r)}{r^2}.$$.	0.2 pts
Total	0.5 pts

Question C.2

Answer	Marks
Using the ideal gas law $P = n\,kT$ where $n = N/V$ where n is the number density, we have $$\frac{dP}{dr} = kT\frac{dn(r)}{dr} + kn(r)\frac{dT}{dr} = -n(r)m_p\frac{Gm'(r)}{r^2}$$ Thus we have $$m'(r) = -\frac{kT}{Gm_p}\left(\frac{r^2}{n(r)}\frac{dn(r)}{dr} + \frac{r^2}{T(r)}\frac{dT(r)}{dr}\right).$$	0.5 pts
Total	0.5 pts

Question C.3

Answer	Marks
If we have isothermal distribution, we have $dT/dr = 0$ and $$m'(r) = -\frac{kT_0}{Gm_p}\left(\frac{r^2}{n(r)}\frac{dn(r)}{dr}\right)$$	0.2 pts
From information about interstellar gas number density, we have $$\frac{1}{n(r)}\frac{dn(r)}{dr} = -\frac{3r+\beta}{r(r+\beta)}$$ Thus we have $$m'(r) = \frac{kT_0 r}{Gm_p}\frac{3r+\beta}{(r+\beta)}$$	0.2 pts

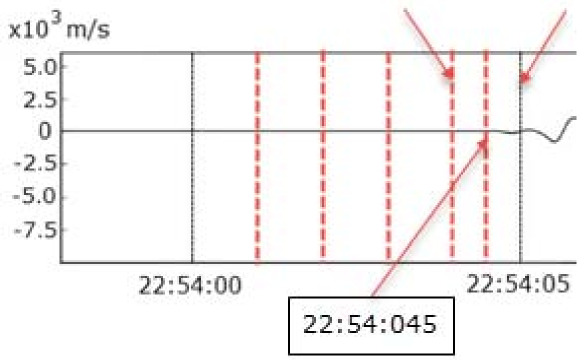
Mass density of the interstellar gas is $$\rho_g(r) = \frac{\alpha m_p}{r(\beta + r)^2}$$ Thus $$m'(r) = \int_0^r (\rho_g(r') + \rho_{dm}(r')) 4\pi r'^2 \, dr' = \frac{kT_0 r}{Gm_p} \frac{3r+\beta}{(r+\beta)}$$ $$m'(r) = \int_0^r \left(\frac{\alpha m_p}{r'(\beta + r')^2} + \rho_{dm}(r') \right) 4\pi r'^2 \, dr' = \frac{kT_0 r}{Gm_p} \frac{3r+\beta}{(r+\beta)}$$	0.3 pts
$$\left(\frac{\alpha m_p}{r(\beta + r)^2} + \rho_{dm}(r) \right) 4\pi r^2 = \frac{kT_0}{Gm_p} \frac{3r^2 + 6r\beta + \beta^2}{(r+\beta)^2}$$ $$\rho_{dm}(r) = \frac{kT_0}{4\pi G m_p} \frac{3r^2 + 6r\beta + \beta^2}{(r+\beta)^2 r^2} - \frac{\alpha m_p}{r(\beta + r)^2}$$	0.3 pts
Total	1.0 pts

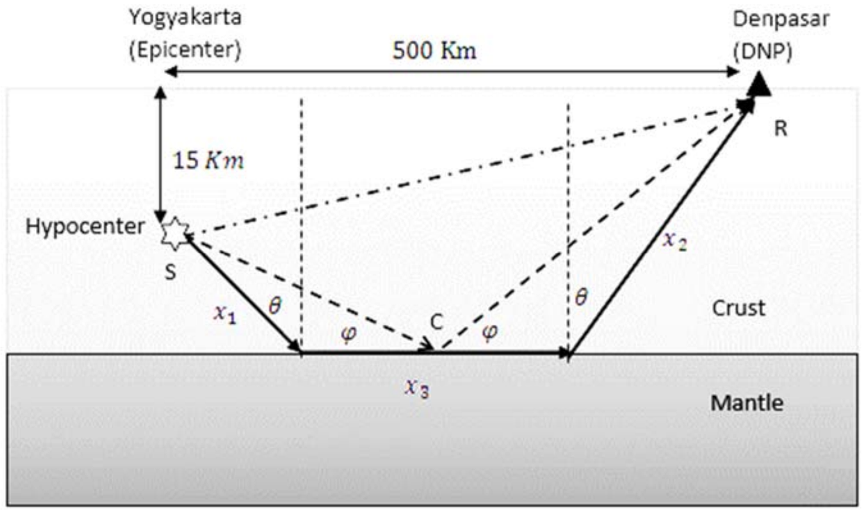
Earthquake, Volcano and Tsunami

A. Merapi Volcano Eruption

Question	Answer	Marks
A.1	Using Black's Principle the equilibrium temperature can be obtained $$m_w c_{vw}(T_e - T_w) + m_m c_{vm}(T_e - T_m) = 0$$ Thus, $$T_e = \frac{m_w c_{vw} T_w + m_m c_{vm} T_m}{m_w c_{vw} + m_m c_{vm}}$$	0.5 pts
A.2	For ideal gas, $p_e v_e = RT_e$, thus $$p_e = \frac{R}{v_e} \frac{m_w c_{vw} T_w + m_m c_{vm} T_m}{m_w c_{vw} + m_m c_{vm}}$$	0.3 pts
A.3	The relative velocity u_{rel} can be expressed as $$u_{rel} = \kappa\, p^{\alpha} V^{\beta} m^{\gamma}$$ where κ is a dimensionless constant. Using dimensional analysis, one can obtain that $$LT^{-1} = M^{\alpha+\gamma} L^{-\alpha+3\beta} T^{-2\alpha}$$ $$\alpha + \gamma = 0$$ $$-\alpha + 3\beta = 1$$ $$-2\alpha = -1$$ Therefore $$u_{rel} = \kappa\, p^{1/2} V^{1/2} m^{-1/2}$$	0.5 pts
	Total score	1.3 pts

B. The Yogyakarta Earthquake

Question	Answer	Marks	
B.1	From the given seismogram, fig. 2  One can see that the P-wave arrived at 22:54:045 or (4.5 – 5.5) seconds after the earthquake occurred at the hypocenter.	0.3 pts	0.5 pts
	Since the horizontal distance from the epicenter to the seismic station in Gamping is 22.5 km, and the depth of the hypocenter is 15 km, the distance from the hypocenter to the station is $\sqrt{22.5^2+15^2}\ \text{km}=27.04\ \text{km}$	0.1 pts	
	Therefore, the P-wave velocity is $v_P = \frac{27.04\ \text{Km}}{4.7\ \text{s}} = 5.75\ \text{Km/s}$	0.1 pts	

Question	Answer	Marks	
B.2	Direct wave: $t_{\text{direct}} = \frac{SR}{v_1} = \frac{\sqrt{500^2 + 15^2}}{v_1} = \frac{502.021}{5.753}\ \text{s} = 86.9\ \text{s}$	0.2 pts	0.6 pts
	As in the case of an optical wave, the Snell's law is also applicable to the seismic wave.  Illustration for the traveling seismic Wave Reflected wave: $t_{\text{reflected}} = \frac{SC}{v_1} + \frac{CR}{v_1}$ $SC\cos\varphi + CR\cos\varphi = 500 \Rightarrow \cot\varphi = \frac{500}{45}$ $t_{\text{reflected}} = \frac{45}{v_1 \sin\varphi} = 87.3\ \text{s}$	0.4 pts	

<table>
<tr><th>Question</th><th>Answer</th><th colspan="2">Marks</th></tr>
<tr><td rowspan="3">B.3</td><td>

Velocity of P-wave on the mantle. The fastest wave crossing the mantle is that propagating along the upperpart of the mantle. From the figure on refracted wave, we obtain that

$$\frac{\sin\theta}{v_1}=\frac{1}{v_2};\qquad \sin\theta=\frac{v_1}{v_2};\qquad \cos\theta=\sqrt{1-\left(\frac{v_1}{v_2}\right)^2}$$

$$\cos\theta=\frac{15}{x_1};\quad x_1=\frac{15}{\cos\theta}\ \text{km};\quad x_2=\frac{30}{\cos\theta}\ \text{km}$$

$$x_3=500-\left(x_1+x_2\right)\sin\theta=500-45\tan\theta$$

</td><td>0.4 pts</td><td rowspan="3">1.2 pts</td></tr>
<tr><td>

The total travel time:

$$t=\frac{x_1+x_2}{v_1}+\frac{x_3}{v_2}=\frac{45}{v_1\cos\theta}+\frac{500}{v_2}-\frac{45\tan\theta}{v_2}$$

$$t\cos\theta=45u_1+500u_2\cos\theta-45u_2\sin\theta$$

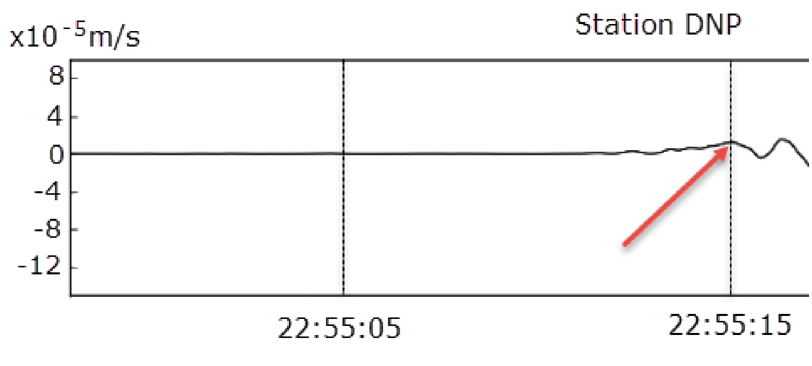
where $u_1=1/v_1$ and $u_2=1/v_2$. Arranging the equation, we get

$$\left(500^2+45^2\right)u_2^2-2t\ 500u_2+t^2-45^2\ u_1=0$$

whose solution is

$$v_2=\frac{500tv_1^2+45v_1\sqrt{\left(45^2+500^2\right)-t^2v_1^2}}{t^2v_1^2-45^2}$$

</td><td>0.5 pts</td></tr>
<tr><td>



From the seismogram, we know that the fastest wave arrived at Denpasar station at 22:55:15, which is $t=75\ \text{s}$ from the origin time of the earthquake in Yogyakarta. Thus

$$v_2=7.1\ \text{km/s}$$

</td><td>0.3 pts</td></tr>
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<table>
<tr><th>Question</th><th>Answer</th><th colspan="2">Marks</th></tr>
<tr><td rowspan="3">B.4</td><td>By using Snell's law and defining $p = \sin\theta / v$ and $u = 1/v$, we obtain

$$p \equiv u(0)\sin\theta_0 = u(z)\sin\theta; \qquad \sin\theta = \frac{p}{u(z)}$$</td><td>0.2 pts</td><td rowspan="3">1.4 pts</td></tr>
<tr><td>where $u(z) = 1/v(z)$ and θ_0 is the initial angle of the seismic wave direction.

$$\frac{dx}{ds} = \sin\theta = \frac{p}{u(z)}; \qquad \frac{dz}{ds} = \cos\theta = \sqrt{1 - \left(\frac{p}{u(z)}\right)^2}$$

$$\frac{dx}{dz} = \frac{dx}{ds}\frac{ds}{dz} = \frac{p}{u}\frac{u}{\left(u^2 - p^2\right)^{1/2}} = p\Big/\left(u^2 - p^2\right)^{1/2}$$

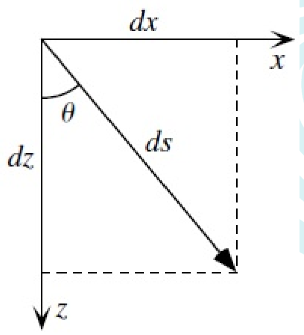
$$x = \int_{z_1}^{z_2} \frac{p}{\left(u^2 - p^2\right)^{1/2}}\, dz$$</td><td>0.5 pts</td></tr>
<tr><td>

Illustration for the direction of wave

The distance *X* is equal to twice the distance from epicenter to the turning point. The turning point is the point when $\theta = 90°$. Thus

$$p = u(z_t) = \frac{1}{v_0 + az_t}; \qquad z_t = \frac{1 - pv_0}{ap}$$

$$X = 2\int_0^{z_t} \frac{p(v_0 + az)}{(1 - p^2(v_0 + az)^2)^{1/2}} dz = \frac{2}{ap}\left(\sqrt{1 - p^2(v_0 + az)^2} - \sqrt{1 - p^2 v_0^2}\right)$$</td><td>0.7 pts</td></tr>
</table>

Question	Answer	Marks	
B.5	For the travel time, $dt = \frac{ds}{v(z)}$; $\quad \frac{dt}{ds} = u(z)$. Thus $$\frac{dt}{dz} = \frac{dt}{ds}\frac{ds}{dz} = \frac{u^2}{(u^2 - p^2)^{1/2}}$$ and therefore $$T = 2\int_0^{z_t} \frac{u^2}{(u^2 - p^2)^{1/2}} dz = 2\int_0^{z_t} \frac{1}{(v_0 + az)} \frac{1}{(1 - p^2 (v_0 + az)^2)^{1/2}} dz$$	1.0 pts	1.0 pts
B.6	The total travel time from the source to the Denpasar can be calculated using previous relation $$T(p) = 2\int_0^{z_t} \frac{u^2(z)}{\left(u^2(z) - p^2\right)^{1/2}} dz$$ Which is valid for a continuous $u(z)$. For a simplified stacked of homogeneous layers (Figure F), the integral equation became a summation $$T(p) = 2\sum_i^N \frac{u_i^2 \Delta z_i}{\left(u_i^2 - p^2\right)^{1/2}}$$	0.6 pts	1.0 pts
	$$\begin{aligned} T(p) &= 2\frac{u_1^2 \Delta z_1}{(u_1^2 - p^2)^{\frac{1}{2}}} + 2\frac{u_2^2 \Delta z_2}{(u_2^2 - p^2)^{\frac{1}{2}}} + 2\frac{u_3^2 \Delta z_3}{(u_3^2 - p^2)^{\frac{1}{2}}} \\ &= \frac{2 \times (0.1504)^2 \times 6}{(0.1504^2 - 0.143^2)^{\frac{1}{2}}} + \frac{2 \times (0.1435)^2 \times 9}{(0.1435^2 - 0.143^2)^{\frac{1}{2}}} \\ &\quad + \frac{2 \times (0.1431)^2 \times 15}{(0.1431^2 - 0.143^2)^{\frac{1}{2}}} \\ &= 151.64 \text{ second} \end{aligned}$$ Note that the actual travel time from the epicenter to Denpasar is 75 seconds. By varying the parameters of velocity and depth up to suitable value of observed travel time, physicist can know Earth structure.	0.4 pts	
		Total score	5.7 pts

C. Java Tsunami

Question	Answer	Marks	
C.1	The center of mass of the raised ocean water with respect to the ocean surface is $h/2$. Thus $$E_P = \frac{h^2 \rho \lambda L g}{4}$$ where ρ is the ocean water density.	0.5 pts	0.5 pts
C.2	Considering a shallow ocean wave in Fig. 5, the whole water (from the surface until the ocean floor) can be considered to be moving due to the wave motion. The potential energy is equal to the kinetic energy. $$\frac{1}{4}\rho \lambda h^2 L g = \frac{1}{4}\rho d L \lambda U^2$$ Where $x = \lambda/2$ and U is the horizontal speed of the water component. The water component that was in the upper part $hL\frac{\lambda}{2}$ should be equal to the one that moves horizontally for a half of period of time $\tau/2$, i.e. $hL\,\lambda/2 = dLU\,\tau/2$. Thus we have $$U = \frac{h\lambda}{\tau d}$$	0.7 pts	1.2 pts
	Accordingly, $\tau = \frac{\lambda}{\sqrt{gd}}$ Thus $$v = \frac{\lambda}{\tau} = \sqrt{gd}$$	0.5 pts	
C.3	Using the argument that the wave energy density is proportional to its amplitude $E = kA^2$ with A is amplitude and k is a proportional constant Because the energy flux is conserve, then $Eva = E_0 v_0 a$ for an area a where the wave flow though. Then, $kA^2\sqrt{gd} = kA_0^2\sqrt{gd_0}$ $$A = A_0\left(\frac{d_0}{d}\right)^{\frac{1}{4}}$$ (Therefore the tsunami wave will increase its amplitude and become narrower as it approaches the beach).	1.3 pts	1.3 pts
		Total score	3.0 pts

Total Score for Problem T2:

Section A :	1.3 points
Section B :	5.7 points
Section C :	3.0 points

Total : 10 points

Cosmic Inflation

A. Expansion of Universe

Question A.1

Answer	Marks
For any test mass m on the boundary of the sphere, $$m\ddot{R}(t) = -GmM_s/R^2(t) \quad \text{(A.1.1)}$$ where M_s is mass portion inside the sphere	**0.2**
Multiplying equation (A.1.1) with $\dot{R}$ and integrating it gives $$\int \dot{R}\frac{d\dot{R}}{dt}\,dt = \frac{1}{2}\,\dot{R}^2 = \frac{GM_s}{R} + A$$ where A is a integration constant	**0.6**
Taking $M_s = \frac{4}{3}\pi R^3(t)\rho(t)$, and $\dot{R} = \dot{a}\,R_s$	**0.2**
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho(t) + \frac{2A}{R_s^2 a^2(t)}$$	**0.2**
Therefore, we have $A_1 = \frac{8\pi G}{3}$	**0.1**
Total	**1.3**

Question A.2

Answer	Marks

The 2nd Friedmann equation can be obtained from the 1st law of thermodynamics : $$dE = -pdV + dQ.$$	**0.1**
For adiabatic processes $dE + pdV = 0$ and its time derivative is $\dot{E} + p\,\dot{V} = 0$.	**0.1**
For the sphere $\dot{V} = V\,(3\,\dot{a}/a)$	**0.1**
Its total energy is $E = \rho(t)V(t)\,c^2$	**0.2**
Therefore $\dot{E} = \left(\dot{\rho} + 3\,\frac{\dot{a}}{a}\right)Vc^2$	**0.1**
It yields $$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$	**0.2**
Therefore, we have $A_2 = 3$.	**0.1**
Total	**0.9**

Question A.3

Answer	Marks
Interpreting $\rho(t)c^2$ as total energy density, and substituting $\frac{p(t)}{c^2} = w\,\rho(t)$ in to the 2nd Friedmann equation yields: $$\dot{\rho} + 3\,\rho(1+w)\frac{\dot{a}}{a} = 0$$	**0.1**
$\rho \propto a^{-3(w+1)}$	**0.2**
(i) In case of radiation, photon as example, the energy is given by $E_r = h\nu = hc/\lambda$ then its energy density $\rho_r = \frac{E_r}{V} \propto a^{-4}$ so that $w_r = \frac{1}{3}$	**0.3**
(ii) In case of nonrelativistic matter, its energy density nearly $\rho_m \simeq \frac{m_0 c^2}{V} \propto a^{-3}$ since dominant energy comes from its rest energy $m_0 c^2$, so that $w_m = 0$	**0.3**
(iii) For a constant energy density, let say $\epsilon_\Lambda =$ constant, $\epsilon_\Lambda \propto a^0$ so that $w_\Lambda = -1$.	**0.3**
Total	**1.2**

Question A.4

Answer	Marks
(i) In case of $k = 0$, for radiation we have $\rho_r a^4 =$ constant. So by comparing the parameters values with their present value, $\rho_r(t)a^4(t) = \rho_{r0}a_0^4$, $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho_{r0}\left(\frac{a_0}{a}\right)^4.$$ $$\int a\,da = \frac{1}{2}a^2 + K = \left(\frac{8\pi G}{3}\,\rho_{r0}a_0^4\right)^{\frac{1}{2}} t.$$	**0.2**
Because $a(t = 0) = 0, K = 0$, then $$a(t) = (2)^{\frac{1}{2}}\left(\frac{8\,\pi G}{3}\,\rho_{r0}a_0^4\right)^{\frac{1}{4}} t^{\frac{1}{2}} = (2H_0)^{\frac{1}{2}}\,t^{\frac{1}{2}}.$$ where $H_0 = \left(\frac{8\pi G}{3}\,\rho_{r0}\right)^{\frac{1}{2}}$ after taking $a_0 = 1$.	**0.2**
(ii) for non-relativistic matter domination, using $\rho_m(t)a^3(t) = \rho_{m0}a_0^3$, and similar way we will get $$a(t) = \left(\frac{3}{2}\right)^{\frac{2}{3}}\left(\frac{8\,\pi G}{3}\,\rho_{m0}a_0^4\right)^{\frac{1}{3}} t^{\frac{2}{3}} = \left(\frac{3H_0}{2}\right)^{\frac{2}{3}}\,t^{\frac{2}{3}}.$$ where $H_0 = \left(\frac{8\pi G}{3}\,\rho_{m0}\right)^{\frac{1}{2}}$.	**0.4**
(iii) for constant energy density, $$\ln a = H_0 t + K'$$ Where K' is integration constant and $H_0 = \left(\frac{8\,\pi G}{3}\,\rho_\Lambda\right)^{\frac{1}{2}}$. Taking condition $a_0 = 1$, $$\ln\left(\frac{a}{a_0}\right) = H_0(t - t_0)$$ $$a(t) = e^{H_0(t-t_0)}$$	**0.4**
Total	**1.2**

Question A.5

Answer	**Marks**
Condition for critical energy condition: $$\rho_c(t) = \frac{3H^2}{8\pi G}$$ Friedmann equation can be written as $$H^2(t) = H^2(t)\Omega(t) - \frac{kc^2}{R_0^2 a^2(t)}$$ $$\left(\frac{R_0^2}{c^2}\right) a^2 H^2(\Omega - 1) = k \qquad \text{(A.5.1)}$$	**0.1**
Total	**0.1**

Question A.6

Answer	Marks
Because $\left(\frac{R_0^2}{c^2}\right) a^2 H^2 > 0$, then $k = +1$ corresponds to $\Omega > 1$, $k = -1$ corresponds to $\Omega < 1$ and $k = 0$ corresponds to $\Omega = 1$	**0.3**
Total	**0.3**

B. Motivation To Introduce Inflation Phase and Its General Conditions

Question B.1

Answer	Marks
Equation (A.5.1) shows that $(\Omega - 1) = \frac{kc^2}{R_0^2}\frac{1}{\dot{a}^2}$.	**0.1**
In a universe dominated by non-relativistic matter or radiation, scale factor can be written as a function of time as $a = a_0\left(\frac{t}{t_0}\right)^p$ where $p < 1$ ($p = \frac{1}{2}$ for radiation and $p = \frac{2}{3}$ for non-relativistic matter)	**0.2**
$(\Omega - 1) = \tilde{k}\, t^{2(1-p)}$	**0.2**
Total	**0.5**

Question B.2

Answer	Marks
For a period dominated by constant energy provides the solution $a(t) = e^{Ht}$ so that $\dot{a} = He^{Ht}$	**0.1**
$(\Omega - 1) = \frac{k}{H^2}\, t^{-2Ht}$	**0.2**
Total	**0.3**

Question B.3

Answer	Marks
Inflation period can be generated by constant energy period, therefore it is a phase where $w = -1$ so that $p = w\rho c^2 = -\rho c^2$ (negative pressure).	**0.2**
Differentiating Friedmann equation leads to $$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^2}$$ $$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2\rho a\,\dot{a}) = \frac{8\pi G}{3}\left(-3\left(\rho + \frac{p}{c^2}\right)a\dot{a} + 2\rho a\dot{a}\right).$$ $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$	**0.4**
So that because during inflation $p = -\rho c^2$, it is equivalent with condition $\ddot{a} > 0$ (accelerated expansion)	**0.1**
As a result, $\ddot{a} = d(\dot{a})/dt = d(Ha)/dt > 0$ or $d(Ha)^{-1}/dt < 0$ (shrinking Hubble radius).	**0.2**
Total	**0.9**

Question B.4

Answer	Marks
Inflation condition can be written as $\frac{d(aH)^{-1}}{dt} < 0$, with $H = \dot{a}/a$ as such $$\frac{d(aH)^{-1}}{dt} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\epsilon) < 0 \Rightarrow \epsilon < 1$$	**0.2**
Total	**0.2**

C. Inflation Generated by Homogenously Distributed Matter

Question C.1

Answer	Marks
Differentiating equations (4) and employing equation 4 we can get $$2H\dot{H} = \frac{1}{3M_{pl}^2}\left[\dot{\phi}\ddot{\phi} + \left(\frac{\partial V}{\partial \phi}\right)\dot{\phi}\right] = \frac{1}{3M_{pl}^2}\left[-3H\,\dot{\phi}^2\right]$$ $$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2}$$	**0.3**
Therefore $\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2 H^2}$	**0.1**
The inflation can occur when the potential energy dominates the particle's energy ($\dot{\phi}^2 \ll V$) such that $H^2 \approx V/(3M_{pl}^2)$.	**0.2**
Slow-roll approximation: $3H\dot{\phi} \approx -V'$	**0.1**
Implies $$\epsilon \approx \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2 \qquad \text{(C.1.1)}$$	**0.3**
we also have $$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi}$$ $\delta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{V''}{3H^2} - \epsilon$ Therefore $$\eta_V \approx M_{Pl}^2 \frac{V''}{V} \qquad \text{(C.1.2)}$$	**0.4**
$$dN = H\,dt = \left(\frac{H}{\dot{\phi}}\right)d\phi \approx -\frac{1}{M_{pl}^2}(V/V')\,d\phi \qquad \text{(C.1.3)}$$ $$\frac{dN}{d\phi} \approx -\frac{1}{M_{pl}^2}(V/V')$$	**0.3**
Total	**1.7**

D. **Inflation with A Simple Potential**

Question D.1

Answer	Marks
Inflation ends at $\epsilon = 1$. Using $V(\phi) = \Lambda^4 \left(\phi/M_{pl}\right)^n$ yields $$\epsilon = \frac{M_{pl}^2}{2}\left[\frac{n}{\phi_{\rm end}}\right]^2 = 1 \implies \phi_{end} = \frac{n}{\sqrt{2}} M_{pl}$$	0.5
Total	0.5

Question D.2

Answer	Marks
From equations (C.1.1), (C.1.2) and (C.1.3) we can obtain $$N = -\left[\frac{\phi}{M_{pl}}\right]^2 \frac{1}{2n} + \beta$$ where β is a integration constant. As $N = 0$ at ϕ_{end} then $\beta = \frac{n}{4}$. $$N = -\left[\frac{\phi}{M_{pl}}\right]^2 \frac{1}{2n} + \frac{n}{4}$$	0.2
$$\eta_V = n(n-1)\left[\frac{M_{pl}}{\phi}\right]^2 = \frac{2(n-1)}{n-4N}$$	0.2
$$\varepsilon = \frac{n^2}{2}\left[\frac{M_{pl}}{\phi}\right]^2 = \frac{n}{n-4N}$$	0.2
so that $$r = 16\varepsilon = \frac{16n}{n-4N}$$	0.1

$$n_s = 1 + 2\eta_V - 6\epsilon = 1 - \frac{2(n+2)}{(n-4N)}$$	**0.1**
To obtain the observational constraint $n_s = 0.968$ we need $n = -5.93$ which is inconsistent with the condition $r < 0.12$. There is no a closest integer n that can obtains $r < 0.12$. As example, for $n = -6$ leads a contradiction $0 < (-0.27)$ and for $n = -5$ leads a contradiction $0 < (-0.2)$.	**0.1**
Total	**0.9**