

PROBLEM

Problem 1



Problem T1. Focus on sketches (13 points)

Part A. Ballistics (4.5 points)

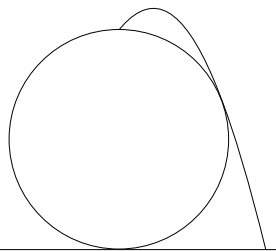
i. (0.8 pts) When the stone is thrown vertically upwards, it can reach the point $x = 0$, $z = v_0^2/2g$ (as it follows from the energy conservation law). Comparing this with the inequality $z \leq z_0 - kx^2$ we conclude that

$$z_0 = v_0^2/2g. \quad [0.3 \text{ pts}]$$

Let us consider the asymptotics $z \rightarrow -\infty$; the trajectory of the stone is a parabola, and at this limit, the horizontal displacement (for the given z) is very sensitive with respect to the curvature of the parabola: the flatter the parabola, the larger the displacement. The parabola has the flattest shape when the stone is thrown horizontally, $x = v_0 t$ and $z = -gt^2/2$, i.e. its trajectory is given by $z = -gx^2/2v_0^2$. Now, let us recall that $z \leq z_0 - kx^2$, i.e. $-gx^2/2v_0^2 \leq z_0 - kx^2 \Rightarrow k \leq g/2v_0^2$. Note that $k < g/2v_0^2$ would imply that there is a gap between the parabolic region $z \leq z_0 - kx^2$ and the given trajectory $z = -gx^2/2v_0^2$. This trajectory is supposed to be optimal for hitting targets far below ($z \rightarrow -\infty$), so there should be no such a gap, and hence, we can exclude the option $k < g/2v_0^2$. This leaves us with

$$k = g/2v_0^2. \quad [0.5 \text{ pts}]$$

ii. (1.2 pts) Let us note that the stone trajectory is reversible and due to the energy conservation law, one can equivalently ask, what is the minimal initial speed needed for a stone to be thrown from the topmost point of the spherical building down to the ground without hitting the roof, and what is the respective trajectory. It is easy to understand that the trajectory either needs to touch the roof, or start horizontally from the topmost point with the curvature radius equal to R . Indeed, if neither were the case, it would be possible to keep the same throwing angle and just reduce the speed a little bit — the stone would still reach the ground without hitting the roof. Further, if it were tangent at the topmost point, the trajectory wouldn't touch nor intersect the roof anywhere else, because the curvature of the parabola has maximum at its topmost point. Then, it would be possible to keep the initial speed constant, and increase slightly the throwing angle (from horizontal to slightly upwards): the new trajectory wouldn't be neither tangent at the top nor touch the roof at any other point; now we can reduce the initial speed as we argued previously. So we conclude that the optimal trajectory needs to touch the roof somewhere, as shown in Fig.



iii. (2.5 pts) The brute force approach would be writing down the condition that the optimal trajectory intersects with the building at two points and touches at one. This would be described by a fourth order algebraic equation and therefore, it is not realistic to accomplish such a solution within a reasonable time frame.

Note that the interior of the building needs to lie inside the region where the targets can be hit with a stone thrown from the top with initial speed v_{\min} . Indeed, if we can throw over the building, we can hit anything inside by lowering the throwing angle. On the other hand, the boundary of the targetable region needs to touch the building. Indeed, if there were a gap, it would be possible to hit a target just above the point where the optimal trajectory touches the building; the trajectory through that target wouldn't touch the building anywhere, hence we arrive at a contradiction.

So, with v_0 corresponding to the optimal trajectory, the targetable region touches the building; due to symmetry, overall there are two touching points (for smaller speeds, there would be four, and for larger speeds, there would be none). With the origin at the top of the building, the intersection points are defined by the following system of equations:

$$x^2 + z^2 + 2zR = 0, \quad z = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}.$$

Upon eliminating z , this becomes a biquadratic equation for x :

$$x^4 \left(\frac{g}{2v_0^2} \right)^2 + x^2 \left(\frac{1}{2} - \frac{gR}{v_0^2} \right) + \left(\frac{v_0^2}{4g} + R \right) \frac{v_0^2}{g} = 0.$$

Hence the speed by which the real-valued solutions disappear can be found from the condition that the discriminant vanishes:

$$\left(\frac{1}{2} - \frac{gR}{v_0^2} \right)^2 = \frac{1}{4} + \frac{gR}{v_0^2} \implies \frac{gR}{v_0^2} = 2.$$

Bearing in mind that due to the energy conservation law, at the ground level the squared speed is increased by $4gR$. Thus we finally obtain

$$v_{\min} = \sqrt{v_0^2 + 4gR} = 3\sqrt{\frac{gR}{2}}.$$

Alternative solution using the fact that if a ball is thrown from a point A to a point B (possibly at different heights) with the minimal required launching speed, the initial and terminal velocities are equal. This fact may be known to some of the contestants, but it can be also easily derived. Indeed, suppose that when starting with velocity \vec{v}_0 at point A , the ball will hit after time τ the point B , and $|\vec{v}_0|$ is the minimal speed by which hitting is possible. Now, let us rotate the vector \vec{v}_0 slightly by adding to it a perpendicular small vector $\vec{u} \perp \vec{v}_0$ ($|\vec{u}| \ll |\vec{v}_0|$). With the launching velocity $\vec{v}_1 = \vec{v}_0 + \vec{u}$, the trajectory of the ball will still go almost through point B : near the point B , the displacement of the trajectory cannot change linearly with $|\vec{u}|$. Indeed, a linear in $|\vec{u}|$ displacement would mean that with essentially the same speed $|\vec{v}_1| \approx |\vec{v}_0|$, it would be possible to throw over point B , which is in a contradiction with the optimality of the original trajectory. Hence, the displacement vector $\vec{r}_0(\tau) - \vec{r}_1(\tau)$ needs to be parallel to the trajectory, i.e. to the velocity \vec{v}_B of the ball at point B . Here, $\vec{r}_0(t)$ [$\vec{r}_1(t)$] is the radius vector of the ball as a function of time when it was launched with velocity \vec{v}_0 [\vec{v}_1]. In a free-falling system of reference one can easily see that $\vec{r}_0(t) - \vec{r}_1(t) = (\vec{v}_0 - \vec{v}_1)t$. So, $(\vec{v}_0 - \vec{v}_1)\tau = \vec{u}\tau \parallel \vec{v}_B \Rightarrow \vec{u} \parallel \vec{v}_B \Rightarrow \vec{v}_0 \perp \vec{v}_B$.

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We start again throwing a ball from the point O at the top of the building, and notice that for the optimal trajectory, at the point P , where it touches the building, the velocity is perpendicular both to the radius vector QP (where Q denotes the building's centre), and to the launching velocity. Hence, QP and the launching velocity are parallel. Let O be the origin, and let us denote $\angle OQP = \alpha$. Then the trajectory is given by

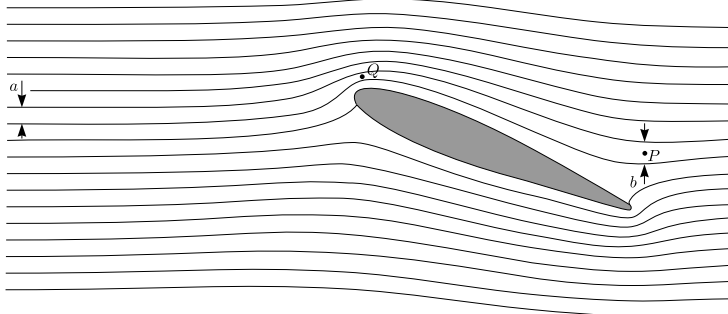
$$z = x \cot \alpha - \frac{gx^2}{2v^2 \sin^2 \alpha}.$$

Point P with coordinates $x = R \sin \alpha$ and $z = R(\cos \alpha - 1)$ belongs to this parabola, hence $R = \frac{gR^2}{2v^2}$, from where we obtain the previous result.

Part B. Mist (4 points)

i. (0.8 pts) In the plane's reference frame, along the channel between two streamlines the volume flux of air (volume flow rate) is constant due to continuity. The volume flux is the product of speed and channel's cross-section area, which, due to the two-dimensional geometry, is proportional to the channel width and can be measured from the Fig. Due to the absence of wind, the unperturbed air's speed in the plane's frame is just v_0 . So, upon measuring the dimensions $a = 10$ mm and $b = 13$ mm (see Fig), we can write $v_0 a = ub$ and hence $u = v_0 \frac{a}{b}$. Since at point P , the streamlines are horizontal where all the velocities are parallel, the vector addition is reduced to the scalar addition: the air's ground speed $v_P = v_0 - u = v_0(1 - \frac{a}{b}) = 23$ m/s.

ii. (1.2 pts) Although the dynamic pressure $\frac{1}{2}\rho v^2$ is relatively small, it gives rise to some adiabatic expansion and compression. In expanding regions the temperature will drop and hence, the pressure of saturated vapours will also drop. If the dew point is reached, a stream of droplets will appear. This process will start in a point where the adiabatic expansion is maximal, i.e. where the hydrostatic pressure is minimal and consequently, as it follows from the Bernoulli's law $p + \frac{1}{2}\rho v^2 = \text{const}$, the dynamic pressure is maximal: in the place where the air speed in wing's frame is maximal and the streamline distance minimal. Such a point Q is marked in Fig.



iii. (2 pts) First we need to calculate the dew point for the air of given water content (since the relative pressure change will be small, we can ignore the dependence of the dew point on pressure). The water vapour pressure is $p_w = p_{sa} r = 2.08$ kPa. The relative change of the pressure of the saturated vapour is small, so we can linearize its temperature dependence:

$$\frac{p_{sa} - p_w}{T_a - T} = \frac{p_{sb} - p_{sa}}{T_b - T_a} \implies T_a - T = (T_b - T_a) \frac{(1-r)p_{sa}}{p_{sb} - p_{sa}};$$

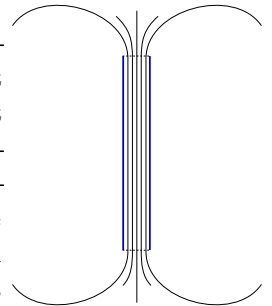
numerically $T \approx 291.5$ K. Further we need to relate the air speed to the temperature. To this end we need to use the energy conservation law. A convenient ready-to-use form of it is provided by the Bernoulli's law. Applying this law will give a good approximation of the reality, but strictly speaking, it needs to be modified to take into account the compressibility of air and the associated expansion/contraction work. Consider one mole of air, which has the mass μ and the volume $V = RT/p$. Apparently the process is fast and the air parcels are large, so that heat transfer across the air parcels is negligible. Additionally, the process is subsonic; all together we can conclude that the process is adiabatic. Consider a segment of a tube formed by the streamlines. Let us denote the physical quantities at its one end by index 1, and at the other end — by index 2. Then, while one mole of gas flows into the tube at one end, as much flows out at the other end. The inflow carries in kinetic energy $\frac{1}{2}\mu v_1^2$, and the outflow carries out $\frac{1}{2}\mu v_2^2$. The inflowing gas receives work due to the pushing gas equal to $p_1 V_1 = RT_1$, the outflowing gas performs work $p_2 V_2 = RT_2$. Let's define molar heat capacities $C_V = \mu c_V$ and $C_p = \mu c_p$. The inflow carries in heat energy $C_V RT_1$, and the outflow carries out $C_V RT_2$. All together, the energy balance can be written as $\frac{1}{2}\mu v^2 + C_p T = \text{const}$. From this we can easily express $\Delta \frac{v^2}{2} = \frac{1}{c} v_{\text{crit}}^2 (\frac{a^2}{c^2} - 1) = c_p \Delta T$, where c is the streamline distance at the point Q , and further

$$v_{\text{crit}} = c \sqrt{\frac{2c_p \Delta T}{a^2 - c^2}} \approx 23 \text{ m/s},$$

where we have used $c \approx 4.5$ mm and $\Delta T = 1.5$ K. Note that in reality, the required speed is probably somewhat higher, because for a fast condensation, a considerable over-saturation is needed. However, within an order of magnitude, this estimate remains valid.

Part C. Magnetic straws (4.5 points)

i. (0.8 pts) Due to the superconducting walls, the magnetic field lines cannot cross the walls, so the flux is constant along the tube. For a closed contour inside the tube, there should be no circulation of the magnetic field, hence the field lines cannot be curved, and the field needs to be homogeneous. The field lines close from outside the tube, similarly to a solenoid.



ii. (1.2 pts) Let us consider the change of the magnetic energy when the tube is stretched (virtually) by a small amount Δl . Note that the magnetic flux through the tube is conserved: any change of flux would imply a non-zero electromotive force $\frac{d\Phi}{dt}$, and for a zero resistivity, an infinite current. So, the induction $B = \frac{\Phi}{\pi r^2}$. The energy density of the magnetic field is $\frac{B^2}{2\mu_0}$. Thus, the change of the magnetic energy is calculated as

$$\Delta W = \frac{B^2}{2\mu_0} \pi r^2 \Delta l = \frac{\Phi^2}{2\mu_0 \pi r^2} \Delta l.$$

This energy increase is achieved owing to the work done by the

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stretching force, $\Delta W = T\Delta l$. Hence, the force

$$T = \frac{\Phi^2}{2\mu_0\pi r^2}.$$

iii. (2.5 pts) Let us analyse, what would be the change of the magnetic energy when one of the straws is displaced to a small distance. The magnetic field inside the tubes will remain constant due to the conservation of magnetic flux, but outside, the magnetic field will be changed. The magnetic field outside the straws is defined by the following condition: there is no circulation of \vec{B} (because there are no currents outside the straws); there are no sources of the field lines, other than the endpoints of the straws; each of the endpoints of the straws is a source of streamlines with a fixed magnetic flux $\pm\Phi$. These are exactly the same condition as those which define the electric field of four charges $\pm Q$. We know that if the distance between charges is much larger than the geometrical size of a charge, the charges can be considered as point charges (the electric field near the charges remains almost constant, so that the respective contribution to the change of the overall electric field energy is negligible). Therefore we can conclude that the endpoints of the straws can be considered as magnetic point charges. In order to calculate the force between two magnetic charges (magnetic monopoles), we need to establish the correspondence between magnetic and electric quantities.

For two electric charges Q separated by a distance a , the force is $F = \frac{1}{4\pi\epsilon_0} \frac{Q^2}{a^2}$, and at the position of one charge, the electric field of the other charge has energy density $w = \frac{1}{32\pi^2\epsilon_0} \frac{Q^2}{a^4}$; hence we can write $F = 8\pi w a^2$. This is a universal expression for the force (for the case when the field lines have the same shape as in the case of two opposite and equal by modulus electric charges) relying only on the energy density, and not related to the nature of the field; so we can apply it to the magnetic field. Indeed, the force can be calculated as a derivative of the full field energy with respect to a virtual displacement of a field line source (electric or magnetic charge); if the energy densities of two fields are respectively equal at one point, they are equal everywhere, and so are equal the full field energies. As it follows from the Gauss law, for a point source of a fixed magnetic flux Φ at a distance a , the induction $B = \frac{1}{4\pi} \frac{\Phi}{a^2}$. So, the energy density $w = \frac{B^2}{2\mu_0} = \frac{1}{32\pi^2\mu_0} \frac{\Phi^2}{a^4}$, hence

$$F = \frac{1}{4\pi\mu_0} \frac{\Phi^2}{a^2}.$$

For the two straws, we have four magnetic charges. The lon-

gitudinal (along a straw axis) forces cancel out (the diagonally positioned pairs of same-sign-charges push in opposite directions). The normal force is a superposition of the attraction due to the two pairs of opposite charges, $F_1 = \frac{1}{4\pi\mu_0} \frac{\Phi^2}{l^2}$, and the repulsive forces of diagonal pairs, $F_2 = \frac{\sqrt{2}}{8\pi\mu_0} \frac{\Phi^2}{2l^2}$. The net attractive force will be

$$F = 2(F_1 - F_2) = \frac{4 - \sqrt{2}}{8\pi\mu_0} \frac{\Phi^2}{l^2}.$$

Alternative solution based on dipole energy calculation.

It is known that the axial component of the magnetic induction created by a solenoidal current of surface density j is proportional to the solid angle Ω under which the current is seen from the given point:

$$B_{\parallel} = \mu_0 j \Omega / 4\pi;$$

this can be easily derived from the Biot-Savart law. Let the distance between the tubes be a (we'll take derivative over a), and let us consider a first tube's point which has a coordinate x (with $0 \leq x \leq l$) from where the direction to the one endpoint of the other tube forms an angle $\alpha = \arctan a/x$ with the tube's axis. From that point, the open circular face of the other tube forms a solid angle $\Omega = \pi r^2 \sin^2 \alpha \cos \alpha / a^2$, so that its contribution to the axial magnetic field at the point x

$$B_{\parallel} = \frac{\mu_0 j r^2 \sin^2 \alpha \cos \alpha}{4a^2} = \frac{\Phi \sin^2 \alpha \cos \alpha}{4\pi a^2}.$$

The solenoidal current at that point forms a magnetic dipole

$$dm = \pi r^2 j dx = \Phi dx / \mu_0,$$

which has potential energy

$$dU = B_{\parallel} dm = \sin^2 \alpha \cos \alpha \Phi^2 dx / 4\pi\mu_0 a^2.$$

When integrating over x , α varies from $\arctan a/l$ to 0, so that

$$U_1 = \int dU = \int \frac{\sin^2 \alpha \cos \alpha \Phi^2 dx}{4\pi\mu_0 a^2} = \int \frac{\cos \alpha \Phi^2 d\alpha}{4\pi\mu_0 a}.$$

Bearing in mind that the other end-circle of the other tube contributes the same amount to the magnetic interaction energy, we find

$$U = 2U_1 = \frac{\Phi^2}{2\pi\mu_0} \left(\frac{1}{a} - \frac{1}{\sqrt{a^2 + l^2}} \right).$$

Upon taking derivative over a and using $a = l$ we obtain the same result for F as previously.

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Problem 2



Problem T2. Kelvin water dropper (8 points)

Part A. Single pipe (4 points)

i. (1.2 pts) Let us write the force balance for the droplet. Since $d \ll r$, we can neglect the force $\frac{\pi}{4}\Delta p d^2$ due to the excess pressure Δp inside the tube. So, the gravity force $\frac{4}{3}\pi r^3 \rho g$ is balanced by the capillary force. When the droplet separates from the tube, the water surface forms in the vicinity of the nozzle a “neck”, which has vertical tangent. In the horizontal cross-section of that “neck”, the capillary force is vertical and can be calculated as $\pi\sigma d$. So,

$$r_{\max} = \sqrt[3]{\frac{3\sigma d}{4\rho g}}.$$

ii. (1.2 pts) Since $d \ll r$, we can neglect the change of the droplet’s capacitance due to the tube. On the one hand, the droplet’s potential is φ ; on the other hand, it is $\frac{1}{4\pi\epsilon_0}\frac{Q}{r}$. So,

$$Q = 4\pi\epsilon_0\varphi r.$$

iii. (1.6 pts) Excess pressure inside the droplet is caused by the capillary pressure $2\sigma/r$ (increases the inside pressure), and by the electrostatic pressure $\frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\epsilon_0\varphi^2/r^2$ (decreases the pressure). So, the sign of the excess pressure will change, if $\frac{1}{2}\epsilon_0\varphi^2/r^2 = 2\sigma/r$, hence

$$\varphi_{\max} = 2\sqrt{\sigma r/\epsilon_0}.$$

The expression for the electrostatic pressure used above can be derived as follows. The electrostatic force acting on a surface charge of density σ and surface area S is given by $F = \sigma S \cdot \bar{E}$, where \bar{E} is the field at the site without the field created by the surface charge element itself. Note that this force is perpendicular to the surface, so F/S can be interpreted as a pressure. The surface charge gives rise to a field drop on the surface equal to $\Delta E = \sigma/\epsilon_0$ (which follows from the Gauss law); inside the droplet, there is no field due to the conductivity of the droplet: $\bar{E} - \frac{1}{2}\Delta E = 0$; outside the droplet, there is field $E = \bar{E} + \frac{1}{2}\Delta E$, therefore $\bar{E} = \frac{1}{2}E = \frac{1}{2}\Delta E$. Bringing everything together, we obtain the expression used above.

Note that alternatively, this expression can be derived by considering a virtual displacement of a capacitor’s surface and comparing the pressure work $p\Delta V$ with the change of the electrostatic field energy $\frac{1}{2}\epsilon_0 E^2 \Delta V$.

Finally, the answer to the question can be also derived from the requirement that the mechanical work dA done for an infinitesimal droplet inflation needs to be zero. From the energy conservation law, $dW + dW_{\text{el}} = \sigma d(4\pi r^2) + \frac{1}{2}\varphi^2_{\max} dC_d$,

where the droplet’s capacitance $C_d = 4\pi\epsilon_0 r$; the electrical work $dW_{\text{el}} = \varphi_{\max} dq = 4\pi\epsilon_0\varphi^2_{\max} dr$. Putting $dW = 0$ we obtain an equation for φ_{\max} , which recovers the earlier result.

Part B. Two pipes (4 points)

i. (1.2 pts) This is basically the same as Part A-ii, except that the surroundings’ potential is that of the surrounding electrode, $-U/2$ (where $U = q/C$ is the capacitor’s voltage) and droplet has the ground potential (0). As it is not defined which electrode is the positive one, opposite sign of the potential may be chosen, if done consistently. Note that since the cylindrical electrode is long, it shields effectively the environment’s (ground, wall, etc) potential. So, relative to its surroundings, the droplet’s potential is $U/2$. Using the result of Part A we obtain

$$Q = 2\pi\epsilon_0 U r_{\max} = 2\pi\epsilon_0 q r_{\max}/C.$$

ii. (1.5 pts) The sign of the droplet’s charge is the same as that of the capacitor’s opposite plate (which is connected to the farther electrode). So, when the droplet falls into the bowl, it will increase the capacitor’s charge by Q :

$$dq = 2\pi\epsilon_0 U r_{\max} dN = 2\pi\epsilon_0 r_{\max} n dt \frac{q}{C},$$

where $dN = n dt$ is the number of droplets which fall during the time dt . This is a simple linear differential equation which is solved easily to obtain

$$q = q_0 e^{\gamma t}, \quad \gamma = \frac{2\pi\epsilon_0 r_{\max} n}{C} = \frac{\pi\epsilon_0 n}{C} \sqrt[3]{\frac{6\sigma d}{\rho g}}.$$

iii. (1.3 pts) The droplets can reach the bowls if their mechanical energy mgH (where m is the droplet’s mass) is large enough to overcome the electrostatic push: The droplet starts at the point where the electric potential is 0, which is the sum of the potential $U/2$, due to the electrode, and of its self-generated potential $-U/2$. Its motion is not affected by the self-generated field, so it needs to fall from the potential $U/2$ down to the potential $-U/2$, resulting in the change of the electrostatic energy equal to $UQ \leq mgH$, where $Q = 2\pi\epsilon_0 U r_{\max}$ (see above). So,

$$U_{\max} = \frac{mgH}{2\pi\epsilon_0 U_{\max} r_{\max}},$$

$$\therefore U_{\max} = \sqrt{\frac{H\sigma d}{2\epsilon_0 r_{\max}}} = \sqrt[6]{\frac{H^3 g \sigma^2 \rho d^2}{6\epsilon_0^3}}.$$

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Problem 3



Problem T3. Protostar formation (9 points)

i. (0.8 pts)

$$T = \text{const} \implies pV = \text{const}$$

$$V \propto r^3$$

$$\therefore p \propto r^{-3} \implies \frac{p(r_1)}{p(r_0)} = 2^3 = 8.$$

ii. (1 pt) During the period considered the pressure is negligible. Therefore the gas is in free fall. By Gauss' theorem and symmetry, the gravitational field at any point in the ball is equivalent to the one generated when all the mass closer to the center is compressed into the center. Moreover, while the ball has not yet shrunk much, the field strength on its surface does not change much either. The acceleration of the outermost layer stays approximately constant. Thus,

$$t \approx \sqrt{\frac{2(r_0 - r_2)}{g}}$$

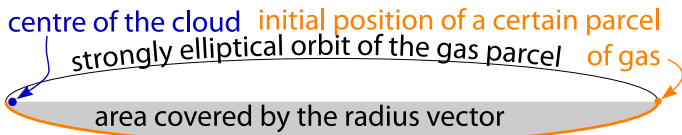
where

$$g \approx \frac{Gm}{r_0^2},$$

$$\therefore t \approx \sqrt{\frac{2r_0^2(r_0 - r_2)}{Gm}} = \sqrt{\frac{0.1r_0^3}{Gm}}.$$

iii. (2.5 pts) Gravitationally the outer layer of the ball is influenced by the rest just as the rest were compressed into a point mass. Therefore we have Keplerian motion: the fall of any part of the outer layer consists in a halfperiod of an ultra-elliptical orbit. The ellipse is degenerate into a line; its foci are at the ends of the line; one focus is at the center of the ball (by Kepler's 1st law) and the other one is at r_0 , see figure (instead of a degenerate ellipse, a strongly elliptical ellipse is depicted). The period of the orbit is determined by the longer semiaxis of the ellipse (by Kepler's 3rd law). The longer semiaxis is $r_0/2$ and we are interested in half a period. Thus, the answer is equal to the halfperiod of a circular orbit of radius $r_0/2$:

$$\left(\frac{2\pi}{2t_{r \rightarrow 0}}\right)^2 \frac{r_0}{2} = \frac{Gm}{(r_0/2)^2} \implies t_{r \rightarrow 0} = \pi \sqrt{\frac{r_0^3}{8Gm}}.$$



Alternatively, one may write the energy conservation law $\frac{\dot{r}^2}{2} - \frac{Gm}{r} = E$ (that in turn is obtainable from Newton's II law $\ddot{r} = -\frac{Gm}{r^2}$ with $E = -\frac{Gm}{r_0}$, separate the variables ($\frac{dr}{dt} = -\sqrt{2E + \frac{2Gm}{r}}$) and write the integral $t = -\int \frac{dr}{\sqrt{2E + \frac{2Gm}{r}}}$. This integral is probably not calculable during the limited time given during the Olympiad, but a possible approach can

be sketched as follows. Substituting $\sqrt{2E + \frac{2Gm}{r}} = \xi$ and $\sqrt{2E} = v$, one gets

$$\frac{t_\infty}{4Gm} = \int_0^\infty \frac{d\xi}{(v^2 - \xi^2)^2}$$

$$= \frac{1}{4v^3} \int_0^\infty \left[\frac{v}{(v-\xi)^2} + \frac{v}{(v+\xi)^2} + \frac{1}{v-\xi} + \frac{1}{v+\xi} \right] d\xi.$$

Here (after shifting the variable) one can use $\int \frac{d\xi}{\xi} = \ln \xi$ and $\int \frac{d\xi}{\xi^2} = -\frac{1}{\xi}$, finally getting the same answer as by Kepler's laws.

iv. (1.7 pts) By Clapeyron–Mendeleev law,

$$p = \frac{mRT_0}{\mu V}.$$

Work done by gravity to compress the ball is

$$W = -\int p dV = -\frac{mRT_0}{\mu} \int_{\frac{4}{3}\pi r_0^3}^{\frac{4}{3}\pi r_3^3} \frac{dV}{V} = \frac{3mRT_0}{\mu} \ln \frac{r_0}{r_3}.$$

The temperature stays constant, so the internal energy does not change; hence, according to the 1st law of thermodynamics, the compression work W is the heat radiated.

v. (1 pt) The collapse continues adiabatically.

$$pV^\gamma = \text{const} \implies TV^{\gamma-1} = \text{const}.$$

$$\therefore T \propto V^{1-\gamma} \propto r^{3-3\gamma}$$

$$\therefore T = T_0 \left(\frac{r_3}{r}\right)^{3\gamma-3}.$$

vi. (2 pts) During the collapse, the gravitational energy is converted into heat. Since $r_3 \gg r_4$, The released gravitational energy can be estimated as $\Delta\Pi = -Gm^2(r_4^{-1} - r_3^{-1}) \approx -Gm^2/r_4$ (exact calculation by integration adds a prefactor $\frac{3}{5}$); the terminal heat energy is estimated as $\Delta Q = c_V \frac{m}{\mu} (T_4 - T_0) \approx c_V \frac{m}{\mu} T_4$ (the approximation $T_4 \gg T_0$ follows from the result of the previous question, when combined with $r_3 \gg r_4$). So, $\Delta Q = \frac{R}{\gamma-1} \frac{m}{\mu} T_4 \approx \frac{m}{\mu} RT_4$. For the temperature T_4 , we can use the result of the previous question, $T_4 = T_0 \left(\frac{r_3}{r_4}\right)^{3\gamma-3}$. Since initial full energy was approximately zero, $\Delta Q + \Delta\Pi \approx 0$, we obtain

$$\frac{Gm^2}{r_4} \approx \frac{m}{\mu} RT_0 \left(\frac{r_3}{r_4}\right)^{3\gamma-3} \implies r_4 \approx r_3 \left(\frac{RT_0 r_3}{\mu m G}\right)^{\frac{1}{3\gamma-4}}.$$

Therefore,

$$T_4 \approx T_0 \left(\frac{RT_0 r_3}{\mu m G}\right)^{\frac{3\gamma-3}{4-3\gamma}}.$$

Alternatively, one can obtain the result by approximately equating the hydrostatic pressure $\rho r_4 \frac{Gm}{r_4^2}$ to the gas pressure $p_4 = \frac{\rho}{\mu} RT_4$; the result will be exactly the same as given above.

The 43rd International Physics Olympiad — July 2012

Grading scheme: Theory

General rules This grading scheme describes the number of points allotted for each term entering a useful formula. These terms don't need to be separately described: if a formula is written correctly, full marks (the sum of the marks of all the terms of that formula) are given. If a formula is not written explicitly, but it is clear that individual terms are written bearing the equation in mind (eg. indicated on a diagram), marks for these terms will be given. Some points are allotted for mathematical calculations.

If a certain term of a useful formula is written incorrectly, 0.1 is subtracted for a minor mistake (eg. missing non-dimensional factor); no mark is given if the mistake is major (with non-matching dimensionality). The same rule is applied to the marking of mathematical calculations: each minor mistake leads to a subtraction of 0.1 pts (as long as the remaining

score for that particular calculation remains positive), and no marks are given in the case of dimensional mistakes.

No penalty is applied in these cases when a mistake is clearly just a rewriting typo (i.e. when there is no mistake in the draft).

If formula is written without deriving: if it is simple enough to be derived in head, full marks, otherwise zero marks.

If there two solutions on Solution sheets, one correct and another incorrect: only the one which corresponds to the Answer Sheets is taken into account. What is crossed out is never considered.

No penalty is applied for propagating errors unless the calculations are significantly simplified (in which case mathematical calculations are credited partially, according to the degree of simplification, with marking granularity of 0.1 pts).

PROBLEM

Problem 1



Problem T1. Focus on sketches (13 points)

Part A. Ballistics (4.5 points)

i. (0.8 pts) If the parameters are derived using the particular cases of throwing up and throwing horizontally:

for throwing up, $z = v_0^2/2$ — 0.2 pts;

from where $z_0 = v_0^2/2g$ — 0.1 pts;

noticing that for horizontally thrown ball,

the trajectory has the same shape as $z = -kx^2$ — 0.2 pts;

finding this trajectory, $z = -gx^2/2v_0^2$ — 0.2 pts;

Concluding $k = g/2v_0^2$ — 0.1 pts;

If, instead of studying the trajectory of a horizontally thrown ball (for which 0.5 pts were allocated), a trajectory of a ball thrown at 45 degrees is studied:

distance is max. when the angle is 45° — 0.2

Finding this maximal distance v^2/g — 0.2

Obtaining k — 0.1

If the parameters are derived using condition that the quadratic equation for the tangent of the throwing angle has exactly one real solution:

Requiring $x = v \cos \alpha t$ — 0.1 pts;

Requiring $z = v \sin \alpha t - gt^2/2$ — 0.2 pts;

Eliminating t : $z = x \tan \alpha - gx^2/v^2 \cos^2 \alpha$ — 0.1 pts;

Obtaining $z - x \tan \alpha + gx^2/v^2(1 + \tan^2 \alpha) = 0$ — 0.1 pts;

Requiring that the discriminant is 0 — 0.2 pts;

Obtaining $z_0 = v_0^2/2g$, $k = gx^2/2v_0^2$ — 0.1 pts;

(If one of the two is incorrect — 0 pts for the last line)

ii. (1.2 pts)

Trajectory hits the sphere when descending — 0.7 pts
(if the top of the parabola is higher than $\frac{3}{2}R$ — 0.5 pts);

Trajectory touches the sphere when ascending — 0.5 pts.

Trajectory touches the sphere at its top or is clearly non-parabolic or **starts inside the sphere or intersects the sphere**: total — 0 pts.

iii. (2.5 pts) If the analysis is based on a trajectory which is wrong in principle, 0 pts.

For minimal speed, $z = z_0 - kx^2$ touches the sphere — 1.5 pts

Shifting the origin to the sphere's top for simplicity — 0.1 pts

Then $z = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$ — 0.2 pts

and $x^2 + z^2 + 2zR = 0$ — 0.2 pts

$\Rightarrow x^4 \left(\frac{g}{2v_0^2}\right)^2 + x^2 \left(\frac{1}{2} - \frac{gR}{v_0^2}\right) + \left(\frac{v_0^2}{4g} + R\right) \frac{v_0^2}{g} = 0$ — 0.1 pts

(full 0.6 pts if equivalent eq. is obtained for non-shifted origin)

Discriminant equals 0 — 0.2 pts

from where $v_0^2 = 0.5gR$ — 0.1 pts

hence $v_{\min} = \sqrt{4.5gR}$ — 0.1 pts

If alternative solution is followed:

Shifting the origin to the sphere's top for simplicity — 0.1 pts

Noting that the touching point vel. is \perp to launching vel. — 0.6 pts

Angle from centre to touching pt P = launching angle — 0.6 pts

Cond. that P belongs to the trajectory — 0.6 pts

From this cond., final answer obtained — 0.6 pts

For a brute force approach:

Obtaining 4th order equation

for intersection points x (or y) — 0.5 pts

which is reduced to cubic (divided by x) — 0.1 pts

Mentioning that it has exactly one pos. root — 0.2 pts
(equivalently, an extremum coincides with a root or there are exactly two distinct real roots.)

Obtaining quadratic eq. for x -coord. of extrema — 0.2 pts

Finding the roots of it — 0.2 pts

Selecting the larger root x_* of it — 0.2 pts

Requiring that x_* is the root of the cubic eq — 0.2 pts

Obtaining the speed as a function of α — 0.2 pts

Finding the minimum of that function — 0.7 pts

Part B. Air flow around a wing (4 points)

i. (0.8 pts)

Using the wing's frame of reference — 0.1 pts

taking streamline distance measurement at P — 0.2 pts
(if off by 1 mm or more, 0.1 pts)

measuring unperturbed streamline distance — 0.1 pts
(if off by 1 mm or more, 0 pts)

at P , streamlines are horiz. \Rightarrow scalar adding — 0.1 pts

Writing continuity condition — 0.2 pts

Finding final answer — 0.1 pts

(if the speed in wing's frame is given as final answer, points are lost for 4th and 6th line)

ii. (1.2 pts)

Writing down continuity condition — 0.2 pts

(or stating: smaller streamline distance \Rightarrow larger speed)

Writing Bernoulli's law — 0.4 pts

0 pts out of 0.4 if the Bernoulli law includes ρgh

(or stating that larger speed \Rightarrow smaller pressure)

Writing adiabatic law — 0.4 pts

PROBLEM

Problem 1



(or stating that smaller pressure \Rightarrow lower temperature)

Finding Q as in the figure in the solutions — 0.2 pts
(if Q marked below the wing's tip — 0.1 pts)

iii. (2 pts)

Finding the dew point: idea of linearization — 0.2 pts

Expression and/or numerical value for dew point — 0.2 pts

Deriving $\frac{1}{2}\mu v^2 + c_p T = \text{const}$:

1 mole of gas carries kin. en. $\frac{1}{2}\mu v^2$ — 0.2 pts

1 mole of gas carries heat en. $C_V T$ — 0.2 pts

work done on 1 mole of gas: $p_1 V_1 - p_2 V_2$ — 0.4 pts

en.: $\frac{1}{2}\mu(v_2^2 - v_1^2) + C_V(T_2 - T_1) = p_1 V_1 - p_2 V_2$ — 0.2 pts

using ideal gas law obtaining $\frac{1}{2}v^2 + c_p T = \text{const}$ — 0.2 pts

Alternative approximate approach:

Bernoulli's law $\frac{1}{2}\rho v^2 + p = \text{const}$ — 0.3 pts

0 pts out of 0.3 if the Bernoulli law includes ρgh

Adiabatic law $pV^\gamma = \text{const}$ — 0.3 pts

$\Rightarrow p^{1-\gamma} T^\gamma = \text{const}$ — 0.2 pts

Approximation $\Delta p = \frac{\gamma}{\gamma-1} \frac{p}{T} \Delta T$ — 0.2 pts

Leading to $\frac{1}{2}v^2 + \frac{R}{\mu} \frac{c_p}{c_p - c_V} T = \text{const}$ — 0.1 pts

Leading to $\frac{1}{2}v^2 + c_p T = \text{const}$ — 0.1 pts

And further (for either approach):

Bringing it to $\Delta \frac{v^2}{2} = \frac{1}{2}v_{\text{crit}}^2 \left(\frac{a^2}{c^2} - 1 \right) = c_p \Delta T$ — 0.1 pts

Measuring a and c — 0.2 pts

Obtaining $v_{\text{crit}} = c \sqrt{\frac{2c_p \Delta T}{a^2 - c^2}} \approx 23 \text{ m/s}$ — 0.1 pts

Part C. Magnetic straws (4.5 points)

i. (0.8 pts)

Lines straight and parallel inside — 0.6 pts

(if not drawn over the entire length, subtract 0.2)

(can be slightly curved close the tube's end)

Lines curve outwards slightly after the exit — 0.2 pts

(if so curved that more than one closed loop on both sides is depicted in Fig, 0 pts out of 0.2)

ii. (1.2 pts)

Expressing induction as $B = \Phi / \pi r^2$ — 0.2 pts

Stating that $w = \frac{B^2}{2\mu_0}$ — 0.2 pts

Idea of using virtual lengthening — 0.2 pts

(Introducing a lengthening Δl is enough)

Expressing $\Delta W = \frac{B^2}{2\mu_0} \pi r^2 \Delta l$ — 0.2 pts

(Same marks if W expressed for entire tube)

Equating $\Delta W = T \Delta l$ — 0.2 pts

Expressing $T = \frac{\Phi^2}{2\mu_0 \pi r^2}$. — 0.2 pts

iii. (2.5 pts)

Idea of using analogy with el. charges — 1 pt

(Sketch with a quadrupole conf. of el. charges is enough)

Finding the force between two magn. charges via matching el. and magn. quantities is worth 1 pts in total, split down as follows:

Expressing el. stat. force via en. density — 0.5 pts

Using the obtained Eq. to obtain $F = \frac{1}{4\pi\mu_0} \frac{\Phi^2}{a^2}$ — 0.5 pts

Any other matching scheme is graded analogously; e.g. finding such a Q which has the same en. density as Φ — 0.5 pts; expressing the force between magn. charges (Φ) as the force between the matching el. charges (Q) — 0.5 pts. Declaring matching pairs $Q \leftrightarrow \Phi$ and $\frac{1}{4\pi\epsilon_0} \leftrightarrow \frac{1}{4\pi\mu_0}$ without energy-based-motivation gives only 0.5 pts.

Noting that \parallel to tubes comp. of force = 0 — 0.2 pts

When expressing $F = 2(F_1 - F_2)$: for factor "2" — 0.1 pts

and for finding F_2 — 0.2 pts

(if wrong sign for F_2 , subtract 0.1)

If alternative solution is followed

Plan to express inter. energy as a function of a

intending to find F as a derivative — 0.2 pts

Calculating $B_{\parallel}(x)$ — 0.8 pts

If estimated without dependance on x , 0.4

considering a tube as an array of dipoles — 0.3 pts

expressing $dm = S j dx$ — 0.3 pts

relating j to Φ — 0.2 pts

Expressing $U = \int B(x) dm$ — 0.4 pts

If estimated as $BSjl$, 0.2

Finding $F = dU/da$ — 0.2 pts

Taking into account the factor "2" — 0.1 pts

PROBLEM

Problem 2



Problem T2. Kelvin water dropper (8 points)

Part A. Single pipe (4 points)

i. (1.2 pts) For the terms entering the force balance of a droplet immediately before separation from the nozzle, the points are given as follows:

$$mg \text{ — 0.2 pts;}$$

$$m = \rho V \text{ — 0.1 pts;}$$

$$V = \frac{4}{3}\pi r_{\max}^3 \text{ — 0.1 pts;}$$

$$\pi\sigma d \text{ — 0.4 pts.}$$

(if geometrically obtained $\cos\alpha$ is included, 0.2 pts)

Force balance equation including these terms — 0.2 pts;

For expressing r_{\max} from the equation — 0.2 pts.

ii. (1.2 pts)

Stating that the droplet's potential is φ — 0.2 pts

(if used correctly, this does not need to be explicitly stated).

Expressing the droplet's potential as $\frac{1}{4\pi\epsilon_0} \frac{Q}{r}$ — 0.8 pts

(without correct sign — 0.6 pts).

Expressing Q from the obtained equation — 0.2 pts.

iii. (1.6 pts) The components of the excess pressure are graded as follows.

$$2\sigma/r \text{ — 0.5 pts;}$$

$$\frac{1}{2}\epsilon_0 E^2 \text{ — 0.4 pts;}$$

bringing it to the form $\frac{1}{2}\epsilon_0\varphi^2/r^2$ — 0.2 pts.

Noticing that the two effects have opposite sign — 0.2 pts

(if used correctly, this does not need to be stated separately).

Equation stating that the excess pressure is 0 — 0.1 pts;

Expressing φ_{\max} from the obtained equation — 0.2 pts.

In the case of energy-balance-based solution, the distribution of marks is as follows.

surface energy change as $4\pi\sigma d(r^2) = 8\pi\sigma r dr$ — 0.5 pts;

electrostatic energy change as $2\pi\epsilon_0\varphi_{\max}^2 dr$ — 0.5 pts;

electrostatic work as $dA_{\text{el}} = 4\pi\epsilon_0\varphi_{\max}^2 dr$ — 0.3 pts;

equation stating that en. change equals to work — 0.1 pts;

expressing φ_{\max} from the obtained equation — 0.2 pts.

0 pts if energy are equated (without virtual displacement).

If factor $\frac{1}{2}$ missing before the electrostat. force (but otherwise correct), -0.2 pts).

Part B. Two pipes (4 points)

i. (1.2 pts)

Stating that the surroundings' potential is $-U/2$ — 0.4 pts;

(if not stated but used correctly — full marks; opposite signs are allowed to be chosen *consistently*)

stating that the droplet's potential is 0 — 0.4 pts

(if not stated but used correctly — full marks)

Applying formula $U = q/C$ — 0.2 pts

Using the result of Part A with

$\varphi = U/2$ to obtain the final result — 0.2 pts.

the solutions with U instead of $U/2$ will qualify for 0.4 pts for the first two lines (i.e. in total up to 0.8)

ii. (1.5 pts)

Stating that a droplet will increase the capacitor's

charge by its own charge $Q = 2\pi\epsilon_0 q r_{\max}/C$ — 0.4 pts;

(0 pts if wrong sign)

(0.2 pts if redundant factor "2")

Expressing $dq = Q dN$ — 0.2 pts;

Substituting $dN = n dt$ — 0.2 pts;

solving the obtained differential equation — 0.5 pts;

determining the integration constant from

the initial condition — 0.1 pts;

substituting r_{\max} from above — 0.1 pts.

iii. (1.3 pts) Equation for the energy balance of a droplet:

expressing the droplet's electrostatic energy change during the fall as UQ — 0.6 pts

(0.3 for $UQ/2$)

expressing the droplet's gravitational energy change as mgH — 0.2 pts

noticing that at the limit voltage, droplet's terminal kinetic energy is zero — 0.3 pts

expressing energy conservation law equation — 0.1 pts

expressing the final answer — 0.1 pts.

(if bowl considered as a point charge, only the mgH line and the kin. en. lines are applicable)

If instead of the energy balance, the force balance is used (which is incorrect), partial credit for the terms entering the equation is given as follows.

gravity force gm — 0.2 pts

electric field estimated as $E \approx U/(H - L/2)$ — 0.2 pts

(0.1, if not realized that this is an approximation)

electric force $F = EQ$ — 0.1 pts

writing down force balance equation — 0.1 pts.

expressing the final answer — 0.1 pts.

(if bowl considered as a point charge, only the first line and the force balance line are applicable)

PROBLEM

Problem 3



Problem T3. Protostar formation (9 points)

i. (0.8 pts)

In thermodynamic equilibrium $T = \text{const}$ — 0.2 pts.
(if not stated but used correctly — full marks)

$$pV = \text{const} \text{ — 0.3 pts.}$$

$$V \propto r^3 \text{ — 0.1 pts.}$$

$$p \propto r^{-3} \text{ — 0.1 pts.}$$

$$\frac{p(r_1)}{p(r_0)} = 8 \text{ — 0.1 pts.}$$

ii. (1 pt)

$$t \approx \sqrt{\frac{2(r_0 - r_2)}{g}} \text{ — 0.4 pts.}$$

$$g \approx \frac{Gm}{r_0^2} \text{ — 0.4 pts.}$$

$$t \approx \sqrt{\frac{0.1r_0^3}{Gm}} \text{ — 0.2 pts.}$$

iii. (2.5 pts)

First solution:

Understanding that we have effectively

interaction of two point masses — 0.5 pts.

(equivalently one may mention the $\propto r^{-2}$ force coming from Gauss' law; if not stated, but used correctly — full marks)

Idea of the motion as an ultraelliptical orbit — 1 pt.

Period of the elliptical orbit is equal to the period
of the circular orbit of the same longer semiaxis — 0.4 pts.
(if not stated but used correctly — full marks)

The longer semiaxis is $r_0/2$ — 0.1 pts.

Equation(s) for the period — 0.3 pts.

We need half a period — 0.1 pts.

$$\text{Final answer } t_{r \rightarrow 0} = \pi \sqrt{\frac{r_0^3}{8Gm}} \text{ — 0.1 pts.}$$

Alternative solution:

Understanding that we have effectively

interaction of two point masses — 0.5 pts.

(equivalently one may mention the $\propto r^{-2}$ force coming from Gauss' law; if not stated, but used correctly — full marks)

Energy conservation as a differential equation — 0.3 pts.

(if only expressed through v (like $\frac{mv^2}{2} - \frac{GMm}{r} = E$) — 0.1 pts.;
if the differential equation of Newton's 2nd law ($\ddot{r} = -\frac{Gm}{r^2}$) is given *instead* — 0.2 pts.)

$$t = \int \frac{dr}{\sqrt{2E + \frac{2Gm}{r}}} \text{ (whichever the sign is) — 0.4 pts.}$$

Integration and final answer — 1.3 pts.

iv. (1.7 pts)

Radiated heat equals the compression work — 1 pt.

[for mentioning or using the 1st law of thermodynamics (possibly in a wrong way) — 0.5 pts.]

$$W = - \int p dV \text{ or } W = - \sum p \Delta V \text{ — 0.3 pts.}$$

(with either “+” or “-” — give full marks)

$$p = \frac{mRT_0}{\mu V} \text{ — 0.2 pts.}$$

$$\text{Calculating the integral, } W = \frac{3mRT_0}{\mu} \ln \frac{r_0}{r_3} \text{ — 0.2 pts.}$$

v. (1 pt)

The collapse continues adiabatically.

(If used, but not written down, give full marks.) — 0.3 pts.

$$pV^\gamma = \text{const} \text{ — 0.3 pts.}$$

$$T \propto V^{1-\gamma} \text{ — 0.2 pts.}$$

$$T = T_0 \left(\frac{r_3}{r} \right)^{3\gamma-3} \text{ — 0.2 pts.}$$

vi. (2 pts)

In case of using energies (marks must not be subtracted for fewer approximations, even in the final answer):

$$T_4 = T_0 \left(\frac{r_3}{r_4} \right)^{3\gamma-3} \text{ — 0.1 pts.}$$

$$\Delta Q + \Delta \Pi \approx 0 \text{ — 0.4 pts.}$$

$$\Delta \Pi \approx -Gm^2/r_4 \text{ — 0.6 pts.}$$

$$\Delta Q = mc_V T_4 \text{ — 0.4 pts.}$$

$$c_V \approx \frac{R}{\mu} \text{ — 0.3 pts.}$$

$$\text{Final answer } r_4 \approx r_3 \left(\frac{RT_0 r_3}{\mu m G} \right)^{\frac{1}{3\gamma-4}} \text{ — 0.1 pts.}$$

$$\text{Final answer } T_4 \approx T_0 \left(\frac{RT_0 r_3}{\mu m G} \right)^{\frac{3\gamma-3}{4-3\gamma}} \text{ — 0.1 pts.}$$

In case of using pressures (again, fewer approximations are permitted):

$$T_4 = T_0 \left(\frac{r_3}{r_4} \right)^{3\gamma-3} \text{ — 0.1 pts.}$$

$$p_4 = p_{\text{hydrostatic}} \text{ — 0.4 pts.}$$

$$p_4 = \frac{\rho}{\mu} RT_4 \text{ — 0.5 pts.}$$

$$p_{\text{hydrostatic}} \approx \rho g r_4 \text{ — 0.4 pts.}$$

$$g \approx \frac{Gm}{r_4^2} \text{ — 0.4 pts.}$$

$$\text{Final answer } r_4 \approx r_3 \left(\frac{RT_0 r_3}{\mu m G} \right)^{\frac{1}{3\gamma-4}} \text{ — 0.1 pts.}$$

$$\text{Final answer } T_4 \approx T_0 \left(\frac{RT_0 r_3}{\mu m G} \right)^{\frac{3\gamma-3}{4-3\gamma}} \text{ — 0.1 pts.}$$