Particles from the Sun[1]

Photons from the surface of the Sun and neutrinos from its core can tell us about solar temperatures and also confirm that the Sun shines because of nuclear reactions.

Throughout this problem, take the mass of the Sun to be $M_\odot = 2.00 \times 10^{30}$ kg, its radius, $R_\odot = 7.00 \times 10^8$ m, its luminosity (radiation energy emitted per unit time), $L_\odot = 3.85 \times 10^{26}$ W, and the Earth-Sun distance, $d_\odot = 1.50 \times 10^{11}$ m.

Note:

(i) $$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax} + \text{constant}$$

(ii) $$\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) e^{ax} + \text{constant}$$

(iii) $$\int x^3 e^{ax} dx = \left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right) e^{ax} + \text{constant}$$

A. Radiation from the Sun :

(A1) Assume that the Sun radiates like a perfect blackbody. Use this fact to calculate the temperature, T_s , of the solar surface. [0.3]

Solution:

Stefan's law: $L_\odot = (4\pi R_\odot^2)(\sigma T_s^4)$

$$T_s = \left(\frac{L_\odot}{4\pi R_\odot^2 \sigma}\right)^{1/4} = 5.76 \times 10^3 \text{ K}$$

The spectrum of solar radiation can be approximated well by the Wien distribution law. Accordingly, the solar energy incident on any surface on the Earth per unit time per unit frequency interval, $u(\nu)$, is given by

$$u(\nu) = A \frac{R_\odot^2}{d_\odot^2} \frac{2\pi h}{c^2} \nu^3 \exp(-h\nu/k_B T_s),$$

where A is the area of the surface normal to the direction of the incident radiation.

Now, consider a solar cell which consists of a thin disc of semiconducting material of area, A, placed perpendicular to the direction of the Sun's rays.

(A2) Using the Wien approximation, express the total power, P_{in}, incident on the surface of the solar cell, in terms of A, $R_\odot$, $d_\odot$, T_s and the fundamental constants c, h, k_B. [0.3]

[1] *Amol Dighe (TIFR), Anwesh Mazumdar (HBCSE-TIFR) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

Solution:

$$P_{\text{in}} = \int_0^\infty u(\nu)d\nu = \int_0^\infty A\,\frac{R_\odot^2}{d_\odot^2}\,\frac{2\pi h}{c^2}\,\nu^3 \exp(-h\nu/k_B T_s)d\nu$$

Let $x = \frac{h\nu}{k_B T_s}$. Then, $\nu = \frac{k_B T_s}{h} x \qquad d\nu = \frac{k_B T_s}{h} dx.$

$$P_{\text{in}} = \frac{2\pi h A R_\odot^2}{c^2 d_\odot^2}\,\frac{(k_B T_s)^4}{h^4}\int_0^\infty x^3 e^{-x}dx = \frac{2\pi k_B^4}{c^2 h^3}\,T_s{}^4\,A\,\frac{R_\odot^2}{d_\odot^2}\cdot 6 = \frac{12\pi k_B^4}{c^2 h^3}\,T_s{}^4\,A\,\frac{R_\odot^2}{d_\odot^2}$$

(A3) Express the number of photons, $n_\gamma(\nu)$, per unit time per unit frequency interval incident on the surface of the solar cell in terms of A, $R_\odot$, $d_\odot$, T_s ν and the fundamental constants c, h, k_B. [0.2]

Solution:

$$\begin{aligned} n_\gamma(\nu) &= \frac{u(\nu)}{h\nu} \\ &= A\,\frac{R_\odot^2}{d_\odot^2}\,\frac{2\pi}{c^2}\,\nu^2 \exp(-h\nu/k_B T_s) \end{aligned}$$

The semiconducting material of the solar cell has a "band gap" of energy, E_g. We assume the following model. Every photon of energy $E \geq E_g$ excites an electron across the band gap. This electron contributes an energy, E_g , as the useful output energy, and any extra energy is dissipated as heat (not converted to useful energy).

(A4) Define $x_g = h\nu_g/k_B T_s$ where $E_g = h\nu_g$. Express the useful output power of the cell, P_{out}, in terms of x_g, A, $R_\odot$, $d_\odot$, T_s and the fundamental constants c, h, k_B. [1.0]

Solution:

The useful power output is the useful energy quantum per photon, $E_g \equiv h\nu_g$, multiplied by the number of photons with energy, $E \geq E_g$.

$$\begin{aligned} P_{\text{out}} &= h\nu_g \int_{\nu_g}^\infty n_\gamma(\nu)d\nu \\ &= h\nu_g\,A\,\frac{R_\odot^2}{d_\odot^2}\,\frac{2\pi}{c^2}\int_{\nu_g}^\infty \nu^2 \exp(-h\nu/k_B T_s)d\nu \\ &= k_B T_s x_g\,A\,\frac{R_\odot^2}{d_\odot^2}\,\frac{2\pi}{c^2}\left(\frac{k_B T_s}{h}\right)^3\int_{x_g}^\infty x^2 e^{-x}dx \\ &= \frac{2\pi k_B^4}{c^2 h^3}\,T_s{}^4\,A\,\frac{R_\odot^2}{d_\odot^2}\,x_g(x_g^2 + 2x_g + 2)e^{-x_g} \end{aligned}$$

(A5) Express the efficiency, η, of this solar cell in terms of x_g. [0.2]

Solution:

Efficiency $\eta = \dfrac{P_{\text{out}}}{P_{\text{in}}} = \dfrac{x_{\text{g}}}{6}\,(x_{\text{g}}^2 + 2x_{\text{g}} + 2)e^{-x_{\text{g}}}$

(A6) Make a qualitative sketch of η versus x_{g}. The values at $x_{\text{g}} = 0$ and $x_{\text{g}} \to \infty$ should be clearly shown. What is the slope of $\eta(x_{\text{g}})$ at $x_{\text{g}} = 0$ and $x_{\text{g}} \to \infty$? [1.0]

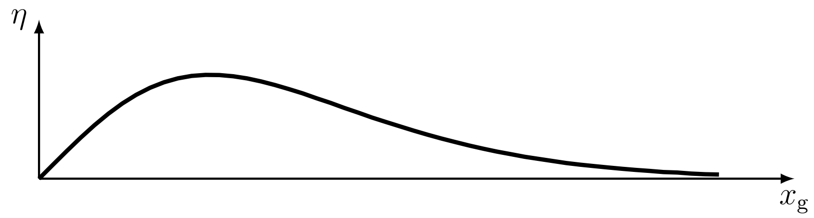
Solution:

$$\eta = \frac{1}{6}\,(x_{\text{g}}^3 + 2x_{\text{g}}^2 + 2x_{\text{g}})e^{-x_{\text{g}}}$$

Put limiting values, $\eta(0) = 0 \qquad \eta(\infty) = 0.$
Since the polynomial has all positive coefficients, it increases monotonically; the exponential function decreases monotonically. Therefore, η has only one maximum.

$$\frac{\mathrm{d}\eta}{\mathrm{d}x_{\text{g}}} = \frac{1}{6}(-x_{\text{g}}^3 + x_{\text{g}}^2 + 2x_{\text{g}} + 2)e^{-x_{\text{g}}}$$

$$\left.\frac{\mathrm{d}\eta}{\mathrm{d}x_{\text{g}}}\right|_{x_{\text{g}}=0} = \frac{1}{3} \qquad \left.\frac{\mathrm{d}\eta}{\mathrm{d}x_{\text{g}}}\right|_{x_{\text{g}}\to\infty} = 0$$



(A7) Let x_0 be the value of x_{g} for which η is maximum. Obtain the cubic equation that gives x_0. Estimate the value of x_0 within an accuracy of ± 0.25. Hence calculate $\eta(x_0)$. [1.0]

Solution:

The maximum will be for $\dfrac{\mathrm{d}\eta}{\mathrm{d}x_{\text{g}}} = \dfrac{1}{6}(-x_{\text{g}}^3 + x_{\text{g}}^2 + 2x_{\text{g}} + 2)e^{-x_{\text{g}}} = 0$

$$\Rightarrow p(x_{\text{g}}) \equiv x_{\text{g}}^3 - x_{\text{g}}^2 - 2x_{\text{g}} - 2 = 0$$

A Numerical Solution by the Bisection Method:

Now,

$$\begin{aligned}
p(0) &= -2 \\
p(1) &= -4 \\
p(2) &= -2 \\
p(3) &= 10 && \Rightarrow \quad 2 < x_0 < 3 \\
p(2.5) &= 2.375 && \Rightarrow \quad 2 < x_0 < 2.5 \\
p(2.25) &= -0.171 && \Rightarrow \quad 2.25 < x_0 < 2.5
\end{aligned}$$

The approximate value of x_{g} where η is maximum is $x_0 = 2.27$.

Alternative methods leading to the same result are acceptable.

$\eta(2.27) = 0.457$

(A8) The band gap of pure silicon is $E_g = 1.11\,\text{eV}$. Calculate the efficiency, η_{Si}, of a silicon solar cell using this value. [0.2]

Solution:

$$x_g = \frac{1.11 \times 1.60 \times 10^{-19}}{1.38 \times 10^{-23} \times 5763} = 2.23$$

$$\eta_{\text{Si}} = \frac{x_g}{6}(x_g^2 + 2x_g + 2)e^{-x_g} = 0.457$$

In the late nineteenth century, Kelvin and Helmholtz (KH) proposed a hypothesis to explain how the Sun shines. They postulated that starting as a very large cloud of matter of mass, $M_\odot$, and negligible density, the Sun has been shrinking continuously. The shining of the Sun would then be due to the release of gravitational energy through this slow contraction.

(A9) Let us assume that the density of matter is uniform inside the Sun. Find the total gravitational potential energy, Ω, of the Sun at present, in terms of G, $M_\odot$ and $R_\odot$. [0.3]

Solution:

The total gravitational potential energy of the Sun: $\Omega = -\int_0^{M_\odot} \frac{Gm\,\mathrm{d}m}{r}$

For constant density, $\rho = \frac{3M_\odot}{4\pi R_\odot^3}$ $\qquad m = \frac{4}{3}\pi r^3 \rho$ $\qquad \mathrm{d}m = 4\pi r^2 \rho \mathrm{d}r$

$$\Omega = -\int_0^{R_\odot} G\left(\frac{4}{3}\pi r^3 \rho\right)(4\pi r^2 \rho)\frac{\mathrm{d}r}{r} = -\frac{16\pi^2 G\rho^2}{3}\frac{R_\odot^5}{5} = -\frac{3}{5}\frac{GM_\odot^2}{R_\odot}$$

(A10) Estimate the maximum possible time τ_{KH} (in years), for which the Sun could have been shining, according to the KH hypothesis. Assume that the luminosity of the Sun has been constant throughout this period. [0.5]

Solution:

$$\tau_{\text{KH}} = \frac{-\Omega}{L_\odot}$$

$$\tau_{\text{KH}} = \frac{3GM_\odot^2}{5R_\odot L_\odot} = 1.88 \times 10^7 \text{years}$$

The τ_{KH} calculated above does not match the age of the solar system estimated from studies of meteorites. This shows that the energy source of the Sun cannot be purely gravitational.

B. Neutrinos from the Sun:

In 1938, Hans Bethe proposed that nuclear fusion of hydrogen into helium in the core of the Sun is the source of its energy. The net nuclear reaction is:

$$4\,{}^{1}\mathrm{H} \longrightarrow {}^{4}\mathrm{He} + 2\mathrm{e}^{+} + 2\nu_{\mathrm{e}}$$

The "electron neutrinos", ν_{e}, produced in this reaction may be taken to be massless. They escape the Sun and their detection on Earth confirms the occurrence of nuclear reactions inside the Sun. Energy carried away by the neutrinos can be neglected in this problem.

(B1) Calculate the flux density, Φ_ν, of the number of neutrinos arriving at the Earth, in units of $\mathrm{m}^{-2}\,\mathrm{s}^{-1}$. The energy released in the above reaction is $\Delta E = 4.0 \times 10^{-12}$ J. Assume that the energy radiated by the Sun is almost entirely due to this reaction. [0.6]

Solution:

$$4.0 \times 10^{-12}\ \mathrm{J} \leftrightarrow 2\nu$$

$$\Rightarrow \Phi_\nu = \frac{L_\odot}{4\pi d_\odot^2\,\delta E} \times 2 = \frac{3.85 \times 10^{26}}{4\pi \times (1.50 \times 10^{11})^2 \times 4.0 \times 10^{-12}} \times 2 = 6.8 \times 10^{14}\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}.$$

Travelling from the core of the Sun to the Earth, some of the electron neutrinos, ν_{e}, are converted to other types of neutrinos, ν_{x}. The efficiency of the detector for detecting ν_{x} is 1/6th of its efficiency for detecting ν_{e}. If there is no neutrino conversion, we expect to detect an average of N_1 neutrinos in a year. However, due to the conversion, an average of N_2 neutrinos (ν_{e} and ν_{x} combined) are actually detected per year.

(B2) In terms of N_1 and N_2, calculate what fraction, f, of ν_{e} is converted to ν_{x}. [0.4]

Solution:

$$\begin{aligned} N_1 &= \epsilon N_0 \\ N_e &= \epsilon N_0(1-f) \\ N_x &= \epsilon N_0 f/6 \\ N_2 &= N_e + N_x \end{aligned}$$

OR

$$(1-f)N_1 + \frac{f}{6}N_1 = N_2$$

$$\Rightarrow f = \frac{6}{5}\left(1 - \frac{N_2}{N_1}\right)$$

In order to detect neutrinos, large detectors filled with water are constructed. Although the interactions of neutrinos with matter are very rare, occasionally they knock out electrons from water molecules in the detector. These energetic electrons move through water at high speeds, emitting electromagnetic radiation in the process. As long as the speed of such an electron is greater than the speed of light in water (refractive index, n), this radiation, called Cherenkov radiation, is emitted in the shape of a cone.

(B3) Assume that an electron knocked out by a neutrino loses energy at a constant rate of α per unit time, while it travels through water. If this electron emits Cherenkov radiation for a time Δt, determine the energy imparted to this electron (E_{imparted}) by the neutrino, in terms of $\alpha, \Delta t, n, m_e, c$. (Assume the electron to be at rest before its interaction with the neutrino.) [2.0]

Solution:

When the electron stops emitting Cherenkov radiation, its speed has reduced to $v_{\text{stop}} = c/n$.

Its total energy at this time is

$$E_{\text{stop}} = \frac{m_e c^2}{\sqrt{1 - v_{\text{stop}}^2/c^2}} = \frac{n m_e c^2}{\sqrt{n^2 - 1}}$$

The energy of the electron when it was knocked out is

$$E_{\text{start}} = \alpha \Delta t + \frac{n m_e c^2}{\sqrt{n^2 - 1}}$$

Before interacting, the energy of the electron was equal to $m_e c^2$.

Thus, the energy imparted by the neutrino is

$$E_{\text{imparted}} = E_{\text{start}} - m_e c^2 = \alpha \Delta t + \left(\frac{n}{\sqrt{n^2 - 1}} - 1 \right) m_e c^2$$

The fusion of H into He inside the Sun takes place in several steps. Nucleus of ^{7}Be (rest mass, m_{Be}) is produced in one of these intermediate steps. Subsequently, it can absorb an electron, producing a ^{7}Li nucleus (rest mass $m_{\text{Li}} < m_{\text{Be}}$) and emitting a ν_e. The corresponding nuclear reaction is:

$$^7\text{Be} + \text{e}^- \longrightarrow {}^7\text{Li} + \nu_e \ .$$

When a Be nucleus ($m_{\text{Be}} = 11.65 \times 10^{-27}$ kg) is at rest and absorbs an electron also at rest, the emitted neutrino has energy $E_\nu = 1.44 \times 10^{-13}$ J. However, the Be nuclei are in random thermal motion due to the temperature T_c at the core of the Sun, and act as moving neutrino sources. As a result, the energy of emitted neutrinos fluctuates with a root mean square value ΔE_{rms}.

(B4) If $\Delta E_{\text{rms}} = 5.54 \times 10^{-17}$ J, calculate the rms speed of the Be nuclei, V_{Be} and hence estimate T_c. (Hint: ΔE_{rms} depends on the rms value of the component of velocity along the line of sight.)

[2.0]

Solution:

Moving ^{7}Be nuclei give rise to Doppler effect for neutrinos. Since the fractional change in energy ($\Delta E_{\rm rms}/E_\nu \sim 10^{-4}$) is small, the Doppler shift may be considered in the non-relativistic limit (a relativistic treatment gives almost same answer). Taking the line of sight along the z-direction,

$$\frac{\Delta E_{\rm rms}}{E_\nu} = \frac{v_{z,rms}}{c}$$
$$= 3.85 \times 10^{-4}$$
$$= \frac{1}{\sqrt{3}}\frac{V_{\rm Be}}{c}$$

$\Rightarrow V_{\rm Be} = \sqrt{3} \times 3.85 \times 10^{-4} \times 3.00 \times 10^{8}\ {\rm m\,s^{-1}} = 2.01 \times 10^{5}\ {\rm m\,s^{-1}}$.

The average temperature is obtained by equating the average kinetic energy to the thermal energy.

$$\frac{1}{2}m_{\rm Be}V_{\rm Be}^2 = \frac{3}{2}k_{\rm B}T_{\rm c}$$
$$\Rightarrow \qquad T_{\rm c} = 1.13 \times 10^{7}\,{\rm K}$$

The Extremum Principle[1]

A. The Extremum Principle in Mechanics

Consider a horizontal frictionless x-y plane shown in Fig. 1. It is divided into two regions, I and II, by a line AB satisfying the equation $x = x_1$. The potential energy of a point particle of mass m in region I is $V = 0$ while it is $V = V_0$ in region II. The particle is sent from the origin O with speed v_1 along a line making an angle θ_1 with the x-axis. It reaches point P in region II traveling with speed v_2 along a line that makes an angle θ_2 with the $x-$axis. Ignore gravity and relativistic effects in this entire task T-2 (all parts).

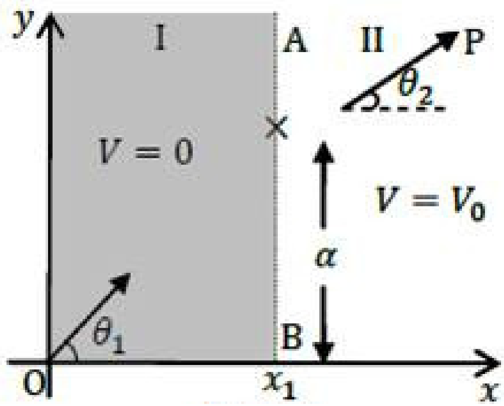


Figure 1

(A1) Obtain an expression for v_2 in terms of m, v_1 and V_0 . [0.2]

Solution:

From the principle of Conservation of Mechanical Energy

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_2^2 + V_0$$

$$v_2 = (v_1^2 - \frac{2V_0}{m})^{1/2}$$

(A2) Express v_2 in terms of v_1, θ_1 and θ_2. [0.3]

Solution:

At the boundary there is an impulsive force ($\propto dV/dx$) in the $-x$ direction. Hence only the velocity component in the $x-$direction v_{1x} suffers change . The component in the $y-$direction remains unchanged. Therefore

$v_{1y} = v_{2y}$

$v_1 \sin\theta_1 = v_2 \sin\theta_2$

We define a quantity called action $A = m\int v(s)\,ds$, where ds is the infinitesimal length along the trajectory of a particle of mass m moving with speed $v(s)$. The integral is taken over the path. As an example. for a particle moving with constant speed v on a circular path of radius R, the action A for one revolution will be $2\pi mRv$. For a particle with constant energy E, it can be shown that of all the possible trajectories between two fixed points, the actual trajectory is the one on which A defined above is an extremum (minimum or maximum). Historically this is known as the Principle of Least Action (PLA).

[1] *Manoj Harbola (IIT-Kanpur) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

(A3) PLA implies that the trajectory of a particle moving between two fixed points in a region of constant potential will be a straight line. Let the two fixed points O and P in Fig. 1 have coordinates (0,0) and (x_0,y_0) respectively and the boundary point where the particle transits from region I to region II have coordinates (x_1,α). Note x_1 is fixed and the action depends on the coordinate α only. State the expression for the action $A(\alpha)$. Use PLA to obtain the the relationship between v_1/v_2 and these coordinates. [1.0]

Solution:

By definition $A(\alpha)$ from O to P is

$$A(\alpha) = mv_1\sqrt{x_1^2+\alpha^2} + mv_2\sqrt{(x_0-x_1)^2+(y_0-\alpha)^2}$$

Differentiating w.r.t. α and setting the derivative of $A(\alpha)$ to zero

$$\frac{v_1\alpha}{(x_1^2+\alpha^2)^{1/2}} - \frac{v_2(y_0-\alpha)}{[(x_0-x_1)^2+(y_0-\alpha)^2]^{1/2}} = 0$$

$$\therefore \frac{v_1}{v_2} = \frac{(y_0-\alpha)\,(x_1^2+\alpha^2)^{1/2}}{\alpha\,[(x_0-x_1)^2+(y_0-\alpha)^2]^{1/2}}$$

Note this is the same as A2, namely $v_1 \sin\theta_1 = v_2 \sin\theta_2$.

B. The Extremum Principle in Optics

A light ray travels from medium I to medium II with refractive indices n_1 and n_2 respectively. The two media are separated by a line parallel to the x-axis. The light ray makes an angle i_1 with the y-axis in medium I and i_2 in medium II (see Fig. 2). To obtain the trajectory of the ray, we make use of another extremum (minimum or maximum) principle known as Fermat's principle of least time.

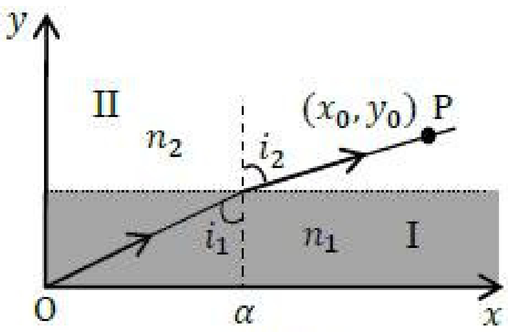


Figure 2

(B1) The principle states that between two fixed points, a light ray moves along a path such that the time taken between the two points is an extremum. Derive the relation between $\sin i_1$ and $\sin i_2$ on the basis of Fermat's principle. [0.5]

Solution:

The speed of light in medium I is c/n_1 and in medium II is c/n_2, where c is the speed of light in vacuum. Let the two media be separated by the fixed line $y = y_1$. Then time $T(\alpha)$ for light to travel from origin (0,0) and (x_0,y_0) is

$$T(\alpha) = n_1(\sqrt{y_1^2+\alpha^2})/c + n_2(\sqrt{(x_0-\alpha)^2+(y_0-y_1)^2})/c$$

Differentiating w.r.t. α and setting the derivative of $T(\alpha)$ to zero

$$\frac{n_1\alpha}{(y_1^2+\alpha^2)^{1/2}} - \frac{n_2(y_0-\alpha)}{[(x_0-\alpha)^2+(y_0-y_1)^2]^{1/2}} = 0$$

$$\therefore \quad n_1 \sin i_1 = n_2 \sin i_2$$

[Note: Derivation is similar to A3. This is Snell's law.]

Shown in Fig. 3 is a schematic sketch of the path of a laser beam incident horizontally on a solution of sugar in which the concentration of sugar decreases with height. As a consequence, the refractive index of the solution also decreases with height.

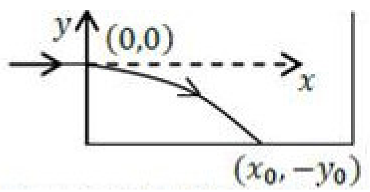


Figure 3

(B2) Assume that the refractive index n(y) depends only on y. Use the equation obtained in B1 to obtain the expresssion for the slope dy/dx of the beam's path in terms of n_0 at $y = 0$ and $n(y)$. [1.5]

Solution:

From Snell's law $n_0 \sin i_0 = n(y) \sin i$

Then, $\quad \dfrac{dy}{dx} = -\cot i$

$$n_0 \sin i_0 = \frac{n(y)}{\sqrt{1+(\frac{dy}{dx})^2}}$$

$$\frac{dy}{dx} = -\sqrt{\left(\frac{n(y)}{n_0 \sin i_0}\right)^2 - 1}$$

(B3) The laser beam is directed horizontally from the origin (0,0) into the sugar solution at a height y_0 from the bottom of the tank as shown. Take $n(y) = n_0 - ky$ where n_0 and k are positive constants. Obtain an expression for x in terms of y and related quantities. You may use: $\int \sec\theta d\theta = \ln(\sec\theta + \tan\theta)$ + constant $\sec\theta = 1/\cos\theta$ or $\int \frac{dx}{\sqrt{x^2-1}} = \ln(x+\sqrt{x^2-1})$ + constant. [1.2]

Solution:

$$\int \frac{dy}{\sqrt{(\frac{n_0-ky}{n_0 \sin\ i_0})^2 - 1}} = -\int dx$$

Note $i_0 = 90^o$ so $\sin i_0 = 1$.

Method I We employ the substitution

$$\xi = \frac{n_0 - ky}{n_0}$$

$$\int \frac{d\xi(-\frac{n_0}{k})}{\sqrt{\xi^2 - 1}} = -\int dx$$

Let $\xi = \sec\theta$. Then

$$\frac{n_0}{k} \ln(\sec\theta + \tan\theta) = x + c$$

Or METHOD II
We employ the substition

$$\xi = \frac{n_0 - ky}{n_0}$$

$$\int \frac{d\xi(-\frac{n_0}{k})}{\sqrt{\xi^2 - 1}} = -\int dx$$

$$-\frac{n_0}{k} \ln\left(\frac{n_0 - ky}{n_0} + \sqrt{(\frac{n_0 - ky}{n_0})^2 - 1}\right) = -x + c$$

Now continuing

Considering the substitutions and boundary condition, $x = 0$ for $y = 0$ we obtain that the constant $c = 0$.
Hence we obtain the following trajectory:

$$x = \frac{n_0}{k} \ln\left(\frac{n_0 - ky}{n_0} + \sqrt{(\frac{n_0 - ky}{n_0})^2 - 1}\right)$$

(B4) Obtain the value of x_0, the point where the beam meets the bottom of the tank. Take y_0 = 10.0 cm, $n_0 = 1.50$, $k = 0.050$ cm^{-1} (1 cm = 10^{-2} m). [0.8]

Solution:

Given $y_0 = 10.0$ cm. $n_0 = 1.50$ $k = 0.050$ cm^{-1}
From (B3)

$$x_0 = \frac{n_0}{k} \ln\left[\left(\frac{n_0 - ky}{n_0}\right) + \left(\left(\frac{n_0 - ky}{n_0}\right)^2 - 1\right)^{1/2}\right]$$

Here $y = -y_0$

$$x_0 = \frac{n_0}{k} \ln \left[\frac{(n_0 + ky_0)}{n_0} + \left(\frac{(n_0 + ky_0)^2}{n_0^2} - 1 \right)^{1/2} \right]$$
$$= 30 \ln \left[\frac{2}{1.5} + \left(\left(\frac{2}{1.5} \right)^2 - 1 \right)^{1/2} \right]$$
$$= 30 \ln \left[\frac{4}{3} + \left(\frac{7}{9} \right)^{1/2} \right]$$
$$= 30 \ln \left[\frac{4}{3} + 0.88 \right]$$
$$= 24.0 \text{ cm}$$

C. The Extremum Principle and the Wave Nature of Matter

We now explore between the PLA and the wave nature of a moving particle. For this we assume that a particle moving from O to P can take all possible trajectories and we will seek a trajectory that depends on the constructive interference of de Broglie waves.

(C1) As the particle moves along its trajectory by an infinitesimal distance Δs, relate the change $\Delta\phi$ in the phase of its de Broglie wave to the change ΔA in the action and the Planck constant. [0.6]

Solution:

From the de Broglie hypothesis

$$\lambda \to \lambda_{dB} = h/mv$$

where λ is the de Broglie wavelength and the other symbols have their usual meaning

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta s$$
$$= \frac{2\pi}{h} mv\Delta s$$
$$= \frac{2\pi \Delta A}{h}$$

(C2) [1.2]

Recall the problem from part A where the particle traverses from O to P (see Fig. 4). Let an opaque partition be placed at the boundary AB between the two regions. There is a small opening CD of width d in AB such that $d \ll (x_0 - x_1)$ and $d \ll x_1$. Consider two extreme paths OCP and ODP such that OCP lies on the classical trajectory discussed in part A. Obtain the phase difference $\Delta\phi_{CD}$ between the two paths to first order.

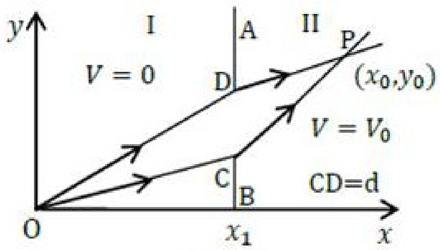
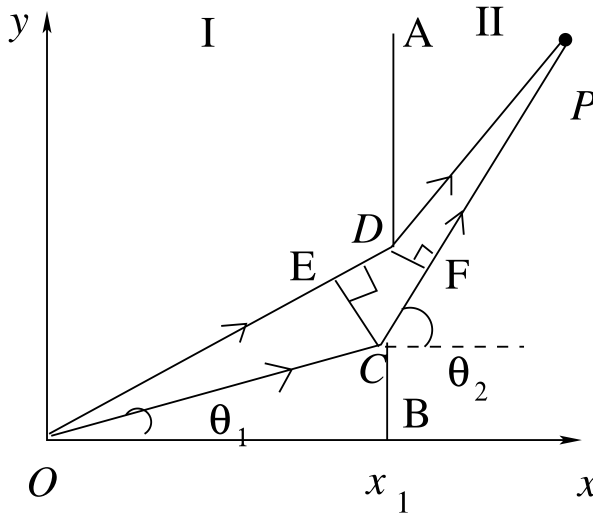


Figure 4

Solution:



Consider the extreme trajectories OCP and ODP of (C1)
The geometrical path difference is ED in region I and CF in region II.
This implies (note: $d \ll (x_0 - x_1)$ and $d \ll x_1$)

$$\Delta\phi_{CD} = \frac{2\pi d \sin\theta_1}{\lambda_1} - \frac{2\pi d \sin\theta_2}{\lambda_2}$$

$$\Delta\phi_{CD} = \frac{2\pi m v_1 d \sin\theta_1}{h} - \frac{2\pi m v_2 d \sin\theta_2}{h}$$
$$= 2\pi \frac{md}{h}(v_1 \sin\theta_1 - v_2 \sin\theta_2)$$
$$= 0 \qquad \text{(from A2 or B1)}$$

Thus near the clasical path there is invariably constructive interference.

D. Matter Wave Interference

Consider an electron gun at O which directs a collimated beam of electrons to a narrow slit at F in the opaque partition A_1B_1 at $x = x_1$ such that OFP is a straight line. P is a point on the screen at $x = x_0$ (see Fig. 5). The speed in I is $v_1 = 2.0000 \times 10^7$ m s^{-1} and $\theta = 10.0000°$. The potential in region II is such that the speed $v_2 = 1.9900 \times 10^7$ m s^{-1}. The distance $x_0 - x_1$ is 250.00 mm (1 mm $= 10^{-3}$ m). Ignore electron-electron interaction.

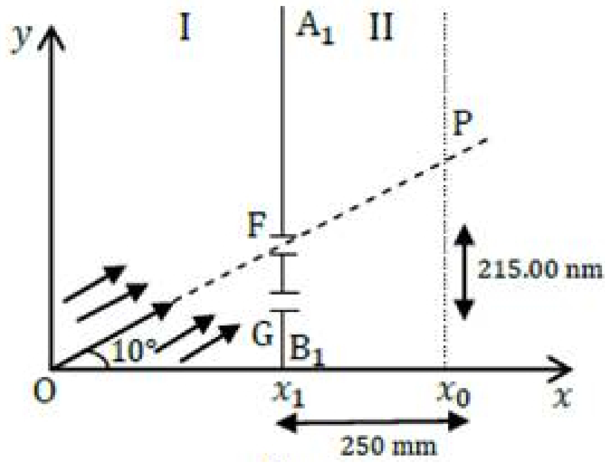


Figure 5

(D1) If the electrons at O have been accelerated from rest, calculate the accelerating potential U_1. [0.3]

Solution:

$$qU_1 = \frac{1}{2}mv^2$$

$$= \frac{9.11 \times 10^{-31} \times 4 \times 10^{14}}{2} J$$

$$= 2 \times 9.11 \times 10^{-17} J$$

$$= \frac{2 \times 9.11 \times 10^{-17}}{1.6 \times 10^{-19}} eV$$

$$= 1.139 \times 10^3 \ eV \quad (\simeq 1100 \ eV)$$

$$U_1 = 1.139 \times 10^3 \ V$$

(D2) Another identical slit G is made in the partition A_1B_1 at a distance of 215.00 nm (1 nm $= 10^{-9}$ m) below slit F (Fig. 5). If the phase difference between de Broglie waves ariving at P from F and G is $2\pi\beta$, calculate β. [0.8]

Solution: Phase difference at P is

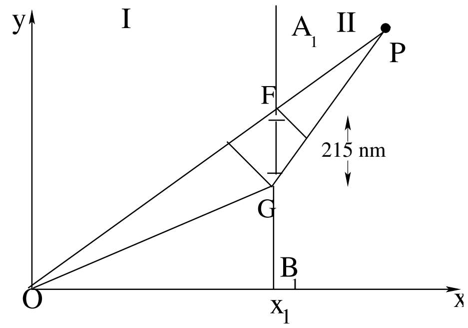
$$\Delta\phi_P = \frac{2\pi d \sin\theta}{\lambda_1} - \frac{2\pi d \sin\theta}{\lambda_2}$$

$$= 2\pi(v_1 - v_2)\frac{md}{h} \sin 10° = 2\pi\beta$$

$$\beta = 5.13$$

(D3) What is is the smallest distance Δy from P at which null (zero) electron detection maybe expected on the screen? [Note: you may find the approximation $\sin(\theta + \Delta\theta) \approx \sin\theta + \Delta\theta\cos\theta$ useful] [1.2]

Solution:



From previous part for null (zero) electron detection $\Delta\phi = 5.5 \times 2\pi$

$$\therefore mv_1 \frac{d\sin\theta}{h} - \frac{mv_2 d\sin(\theta+\Delta\theta)}{h} = 5.5$$

$$\begin{aligned} sin(\theta+\Delta\theta) &= \frac{\frac{mv_1 d\sin\theta}{h} - 5.5}{\frac{mv_2 d}{h}} \\ &= \frac{v_1}{v_2}\sin\theta - \frac{h}{m}\frac{5.5}{v_2 d} \\ &= \frac{2}{1.99}\sin 10^\circ - \frac{5.5}{1374.78 \times 1.99 \times 10^7 \times \times 2.15 \times 10^{-7}} \\ &= 0.174521 - 0.000935 \end{aligned}$$

This yields $\Delta\theta = -0.0036^\circ$

The closest distance to P is

$$\begin{aligned} \Delta y &= (x_0 - x_1)(\tan(\theta+\Delta\theta) - \tan\theta) \\ &= 250(\tan 9.9964 - \tan 10) \\ &= -0.0162mm \\ &= -16.2\mu\,\mathrm{m} \end{aligned}$$

The negative sign means that the closest minimum is below P.

Approximate Calculation for θ and Δy

Using the approximation $\sin(\theta+\Delta\theta) \approx \sin\theta + \Delta\theta\cos\theta$

The phase difference of $5.5 \times 2\pi$ gives

$$mv_1\frac{d\sin 10^\circ}{h} - mv_2\frac{d(\sin 10^\circ + \Delta\theta\cos 10^\circ)}{h} = 5.5$$

From solution of the previous part

$$mv_1\frac{d\sin 10^\circ}{h} - mv_2\frac{dsin10^\circ}{h} = 5.13$$

Therefore

$$mv_2 \frac{d\Delta\theta \cos 10^\circ}{h} = 0.3700$$

This yields $\Delta\theta \approx 0.0036^\circ$

$\Delta y = -0.0162\,\text{mm} = -16.2\mu\text{m}$ as before

(D4) The electron beam has a square cross section of 500 nm × 500 nm and the setup is 2 m long. What should be the minimum beam flux density I_{min} (number of electrons per unit normal area per unit time) if, on an average, there is at least one electron in the setup at a given time? [0.4]

Solution: The product of the speed of the electrons and number of electron per unit volume on an average yields the intensity.

Thus $N = 1 =$ Intensity × Area × Length/ Electron Speed

$= I_{min} \times 0.25 \times 10^{-12} \times 2/2 \times 10^7$

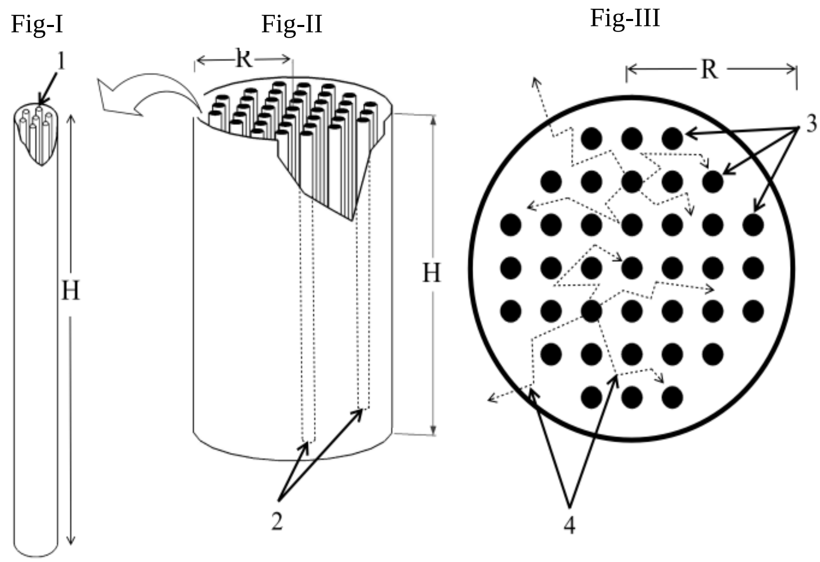
This gives $I_{min} = 4\times\ 10^{19}\ \ \text{m}^{-2}\ \text{s}^{-1}$

R. Bach, D. Pope, Sy-H Liou and H. Batelaan, New J. of Physics Vol. 15, 033018 (2013).

The Design of a Nuclear Reactor[1]

Uranium occurs in nature as UO_2 with only 0.720% of the uranium atoms being ^{235}U. Neutron induced fission occurs readily in ^{235}U with the emission of 2-3 fission neutrons having high kinetic energy. This fission probability will increase if the neutrons inducing fission have low kinetic energy. So by reducing the kinetic energy of the fission neutrons, one can induce a chain of fissions in other ^{235}U nuclei. This forms the basis of the power generating nuclear reactor (NR).

A typical NR consists of a cylindrical tank of height H and radius R filled with a material called moderator. Cylindrical tubes, called fuel channels, each containing a cluster of cylindrical fuel pins of natural UO_2 in solid form of height H, are kept axially in a square array. Fission neutrons, coming outward from a fuel channel, collide with the moderator, losing energy, and reach the surrounding fuel channels with low enough energy to cause fission (Figs I-III). Heat generated from fission in the pin is transmitted to a coolant fluid flowing along its length. In the current problem we shall study some of the physics behind the (A) Fuel Pin, (B) Moderator and (C) NR of cylindrical geometry.



Schematic sketch of the Nuclear Reactor (NR)
Fig-I: Enlarged view of a fuel channel (1-*Fuel Pins*)
Fig-II: A view of the NR (2-*Fuel Channels*)
Fig-III: Top view of NR (3-*Square Arrangement of Fuel Channels* and 4-*Typical Neutron Paths*).
Only components relevant to the problem are shown (e.g. control rods and coolant are not shown).

A. Fuel Pin

Data for UO_2

1. Molecular weight M_w=0.270 kg mol^{-1}
2. Density ρ=1.060×10^4 kg m^{-3}
3. Melting point T_m=3.138×10^3 K
4. Thermal conductivity λ=3.280 W m^{-1} K^{-1}

A1 Consider the following fission reaction of a stationary ^{235}U after it absorbs a neutron of negligible kinetic energy.

$$^{235}\mathrm{U} + {}^{1}\mathrm{n} \longrightarrow {}^{94}\mathrm{Zr} + {}^{140}\mathrm{Ce} + 2\,{}^{1}\mathrm{n} + \Delta E$$

[1] *Joseph Amal Nathan (BARC) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

Estimate ΔE (in MeV) the total fission energy released. The nuclear masses are: $m(^{235}\mathrm{U}) = 235.044$ u; $m(^{94}\mathrm{Zr}) = 93.9063$ u; $m(^{140}\mathrm{Ce}) = 139.905$ u; $m(^{1}\mathrm{n}) = 1.00867$ u and 1 u = 931.502 MeV c^{-2}. Ignore charge imbalance. [0.8]

Solution: $\Delta E = 208.684$ MeV

Detailed solution: The energy released during the transformation is

$$\Delta E = [m(^{235}\mathrm{U}) + m(^{1}\mathrm{n}) - m(^{94}\mathrm{Zr}) - m(^{140}\mathrm{Ce}) - 2m(^{1}\mathrm{n})]\mathrm{c}^2$$

Since the data is supplied in terms of unified atomic masses (u), we have

$$\Delta E = [m(^{235}\mathrm{U}) - m(^{94}\mathrm{Zr}) - m(^{140}\mathrm{Ce}) - m(^{1}\mathrm{n})]\mathrm{c}^2$$

$= 208.684$ MeV [***Acceptable Range* (208.000 to 209.000)**]
from the given data.

A2 Estimate N the number of ^{235}U atoms per unit volume in natural UO_2. [0.5]

Solution: $N = 1.702 \times 10^{26}\ \mathrm{m}^{-3}$

Detailed solution: The number of UO_2 molecules per m^3 of the fuel N_1 is given in the terms of its density ρ, the Avogadro number N_A and the average molecular weight M_w as

$$N_1 = \frac{\rho N_A}{M_w}$$
$$= \frac{10600 \times 6.022 \times 10^{23}}{0.270} = 2.364 \times 10^{28}\ \mathrm{m}^{-3}$$

Each molecule of UO_2 contains one uranium atom. Since only 0.72% of these are ^{235}U,

$N = 0.0072 \times N_1$
$= 1.702 \times 10^{26}\ \mathrm{m}^{-3}$ [***Acceptable Range* (1.650 to 1.750)**]

A3 Assume that the neutron flux $\phi = 2.000 \times 10^{18}\ \mathrm{m}^{-2}\ \mathrm{s}^{-1}$ on the fuel is uniform. The fission cross-section (effective area of the target nucleus) of a ^{235}U nucleus is $\sigma_f = 5.400 \times 10^{-26}$ m^2. If 80.00% of the fission energy is available as heat, estimate Q (in W m^{-3}) the rate of heat production in the pin per unit volume. 1MeV = 1.602 $\times 10^{-13}$ J. [1.2]

Solution: $Q = 4.917 \times 10^8\ \mathrm{W/m}^3$

Detailed solution: It is given that 80% of the fission energy is available as heat thus the heat energy available per fission E_f is from a-(i)

$E_f = 0.8 \times 208.7$ MeV
$= 166.96$ MeV
$= 2.675 \times 10^{-11}$ J

The total cross-section per unit volume is $N \times \sigma_f$. Thus the heat produced per unit

volume per unit time Q is
$Q = N \times \sigma_f \times \phi \times E_f$
$= (1.702 \times 10^{26}) \times (5.4 \times 10^{-26}) \times (2 \times 10^{18}) \times (2.675 \times 10^{-11})$ W/m^3
$= 4.917 \times 10^{8}$ W/m^3 [***Acceptable Range* (4.800 to 5.000)**]

A4 The steady-state temperature difference between the center (T_c) and the surface (T_s) of the pin can be expressed as $T_c - T_s = kF(Q, a, \lambda)$ where $k = 1/4$ is a dimensionless constant and a is the radius of the pin. Obtain $F(Q, a, \lambda)$ by dimensional analysis. [0.5]

Solution: $T_c - T_s = \dfrac{Qa^2}{4\lambda}$.

Detailed solution: The dimensions of $T_c - T_s$ is temperature. We write this as $T_c - T_s = [K]$. Once can similarly write down the dimensions of Q, a and λ. Equating the temperature to powers of Q, a and λ, one could state the following dimensional equation:
$K = Q^\alpha a^\beta \lambda^\gamma$
$= [ML^{-1}T^{-3}]^{\alpha} [L]^{\beta} [ML^{1}T^{-3}K^{-1}]^{\gamma}$
This yields the following algebraic equations
$\gamma =$ -1 equating powers of temperature
$\alpha + \gamma = 0$ equating powers of mass or time. From the previous equation we get $\alpha = 1$
Next $-\alpha + \beta + \gamma = 0$ equating powers of length. This yields $\beta = 2$.
Thus we obtain $T_c - T_s = \dfrac{Qa^2}{4\lambda}$ where we insert the dimensionless factor 1/4 as suggested in the problem. ***No penalty if the factor 1/4 is not written.***

Note: Same credit for alternate ways of obtaining α, β, γ.

A5 The desired temperature of the coolant is 5.770 $\times 10^2$ K. Estimate the upper limit a_u on the radius a of the pin. [1.0]

Solution: $a_u = 8.267 \times 10^{-3}$ m.

Detailed solution: The melting point of UO_2 is 3138 K and the maximum temperature of the coolant is 577 K. This sets a limit on the maximum permissible temperature ($T_c - T_s$) to be less than (3138 - 577 = 2561 K) to avoid "meltdown". Thus one may take a maximum of ($T_c - T_s$) = 2561 K.
Noting that $\lambda = 3.28$ W/m - K, we have

$$a_u^2 = \frac{2561 \times 4 \times 3.28}{4.917 \times 10^8}$$

Where we have used the value of Q from A2. This yields $a_u \backsimeq 8.267 \times 10^{-3}$ m. So $a_u = 8.267 \times 10^{-3}$ m constitutes an upper limit on the radius of the fuel pin.

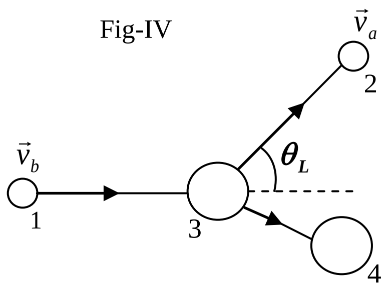
Note: The Tarapur 3 & 4 NR in Western India has a fuel pin radius of 6.090×10^{-3} m.

B. The Moderator

Consider the two dimensional elastic collision between a neutron of mass 1 u and a moderator atom of mass A u. Before collision all the moderator atoms are considered at rest in the laboratory frame (LF). Let $\overrightarrow{v_b}$ and $\overrightarrow{v_a}$ be the velocities of the neutron before and after collision respectively in the LF. Let $\overrightarrow{v_m}$ be the velocity of the center of mass (CM) frame relative to LF and θ be the neutron scattering angle in the CM frame. All the particles involved in collisions are moving at non-relativistic speeds

B1 The collision in LF is shown schematically with θ_L as the scattering angle (Fig-IV). Sketch the collision schematically in CM frame. Label the particle velocities for 1, 2 and 3 in terms of $\overrightarrow{v_b}$, $\overrightarrow{v_a}$ and $\overrightarrow{v_m}$. Indicate the scattering angle θ. [1.0]

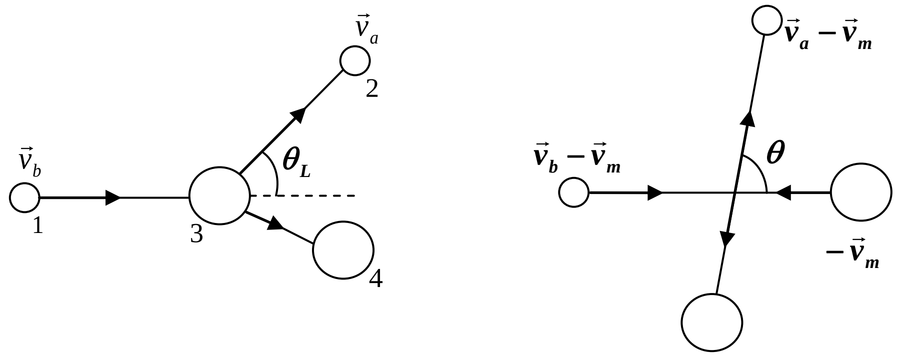
Fig-IV



Collision in the Laboratory Frame
1-*Neutron* before collision
2-*Neutron* after collision
3-*Moderator Atom* before collision
4-*Moderator Atom* after collision

Solution:

Laboratory Frame Center of Mass Frame



B2 Obtain v and V, the speeds of the neutron and the moderator atom in the CM frame after the collision, in terms of A and v_b. [1.0]

Solution: ***<u>Detailed solution</u>*:** Before the collision in the CM frame $(v_b - v_m)$ and v_m will be magnitude of the velocities of the neutron and moderator atom respectively. From momentum conservation in the CM frame, $v_b - v_m = Av_m$ gives $v_m = \frac{v_b}{A+1}$.

After the collision, let v and V be magnitude of the velocities of neutron and moderator atom respectively in the CM frame. From conservation laws,

$$v = AV \qquad \text{and} \qquad \frac{1}{2}(v_b - v_m)^2 + \frac{1}{2}Av_m^2 = \frac{1}{2}v^2 + \frac{1}{2}AV^2 . (\rightarrow [0.2 + 0.2])$$

Solving gives $v = \frac{Av_b}{A+1}$ and $V = \frac{v_b}{A+1}$. **(OR)** From definition of center of mass frame $v_m = \frac{v_b}{A+1}$. Before the collision in the CM frame $v_b - v_m = \frac{Av_b}{A+1}$ and v_m will be magnitude of the velocities of the neutron and moderator atom respectively. In elastic collision the particles are scattered in the opposite direction in the CM frame and so the speeds remain same $v = \frac{Av_b}{A+1}$ and $V = \frac{v_b}{A+1}$ ($\rightarrow$ [0.2 + 0.1]).

Note: Alternative solutions are worked out in the end and will get appropriate weightage.

B3 Derive an expression for $G(\alpha,\theta) = E_a/E_b$, where E_b and E_a are the kinetic energies of the neutron, in the LF, before and after the collision respectively, and $\alpha \equiv [(A-1)/(A+1)]^2$, [1.0]

Solution:

$$G(\alpha,\theta) = \frac{E_a}{E_b} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2} = \frac{1}{2}\left[(1+\alpha) + (1-\alpha)\cos\theta\right].$$

Detailed solution: Since $\vec{v_a} = \vec{v} + \vec{v_m}$, $v_a^2 = v^2 + v_m^2 + 2vv_m\cos\theta$ ($\rightarrow$ [0.3]). Substituting the values of v and v_m, $v_a^2 = \frac{A^2v_b^2}{(A+1)^2} + \frac{v_b^2}{(A+1)^2} + \frac{2Av_b^2}{(A+1)^2}\cos\theta$ ($\rightarrow$ [0.2]), so

$$\frac{v_a^2}{v_b^2} = \frac{E_a}{E_b} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2}.$$

$$G(\alpha,\theta) = \frac{A^2+1}{(A+1)^2} + \frac{2A}{(A+1)^2}\cos\theta = \frac{1}{2}\left[(1+\alpha) + (1-\alpha)\cos\theta\right].$$

Alternate form

$$= 1 - \frac{(1-\alpha)(1-\cos\theta)}{2}.$$

Note: Alternative solutions are worked out in the end and will get appropriate weightage.

B4 Assume that the above expression holds for D_2O molecule. Calculate the maximum possible fractional energy loss $f_l \equiv \frac{E_b - E_a}{E_b}$ of the neutron for the D_2O (20 u) moderator. [0.5]

Solution: $f_l = 0.181$

Detailed solution: The maximum energy loss will be when the collision is head on ie., E_a will be minimum for the scattering angle $\theta = \pi$.

So $E_a = E_{min} = \alpha E_b$.

For D_2O, $\alpha = 0.819$ and maximum fractional loss $\left(\frac{E_b - E_{min}}{E_b}\right) = 1 - \alpha = 0.181$. [***Acceptable Range* (0.170 to 0.190)**]

C. The Nuclear Reactor

To operate the NR at any constant neutron flux Ψ (steady state), the leakage of neutrons has to be compensated by an excess production of neutrons in the reactor. For a reactor in cylindrical geometry the leakage rate is $k_1\left[\left(\frac{2.405}{R}\right)^2+\left(\frac{\pi}{H}\right)^2\right]\Psi$ and the excess production rate is $k_2\Psi$. The constants k_1 and k_2 depend on the material properties of the NR.

C1 Consider a NR with $k_1 = 1.021\times10^{-2}$ m and $k_2 = 8.787\times10^{-3}$ m^{-1}. Noting that for a fixed volume the leakage rate is to be minimized for efficient fuel utilisation obtain the dimensions of the NR in the steady state. [1.5]

Solution: $R = 3.175$ m, $H = 5.866$ m.

Detailed solution: For constant volume $V = \pi R^2 H$,

$$\frac{d}{dH}\left[\left(\frac{2.405}{R}\right)^2+\left(\frac{\pi}{H}\right)^2\right]=0,$$

$$\frac{d}{dH}\left[\frac{2.405^2\pi H}{V}+\frac{\pi^2}{H^2}\right]=\frac{2.405^2\pi}{V}-2\frac{\pi^2}{H^3}=0,$$

gives $\left(\frac{2.405}{R}\right)^2=2\left(\frac{\pi}{H}\right)^2$.

For steady state,

$$1.021\times10^{-2}\left[\left(\frac{2.405}{R}\right)^2+\left(\frac{\pi}{H}\right)^2\right]\Psi=8.787\times10^{-3}\ \Psi.$$

Hence $H = 5.866$ m [***Acceptable Range*** **(5.870 to 5.890)**]
$R = 3.175$ m [***Acceptable Range*** **(3.170 to 3.180)**].

Alternative Non-Calculus Method to Optimize

Minimisation of the expression $\left(\frac{2.405}{R}\right)^2+\left(\frac{\pi}{H}\right)^2$**, for a fixed volume** $V=\pi R^2 H$:

Substituting for R^2 in terms of V, H we get $\frac{2.405^2\pi H}{V}+\frac{\pi^2}{H^2}$,

which can be written as, $\frac{2.405^2\pi H}{2V}+\frac{2.405^2\pi H}{2V}+\frac{\pi^2}{H^2}$.

Since all the terms are positive applying AMGM inequality for three positive terms we get

$$\frac{\frac{2.405^2\pi H}{2V}+\frac{2.405^2\pi H}{2V}+\frac{\pi^2}{H^2}}{3}\geq\sqrt[3]{\frac{2.405^2\pi H}{2V}\times\frac{2.405^2\pi H}{2V}\times\frac{\pi^2}{H^2}}=\sqrt[3]{\frac{2.405^4\pi^4}{4V^2}}.$$

The RHS is a constant. The LHS is always greater or equal to this constant implies that this is the minimum value the LHS can achieve. The minimum is achieved when all the three positive terms are equal, which gives the condition $\frac{2.405^2 \pi H}{2V} = \frac{\pi^2}{H^2} \Rightarrow \left(\frac{2.405}{R}\right)^2 = 2\left(\frac{\pi}{H}\right)^2$.

For steady state,

$$1.021 \times 10^{-2}\left[\left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2\right]\Psi = 8.787 \times 10^{-3}\ \Psi.$$

Hence $H = 5.866$ m [***Acceptable Range* (5.870 to 5.890)**]
$R = 3.175$ m [***Acceptable Range* (3.170 to 3.180)**].

Note: Putting the condition in the RHS gives the minimum as $\frac{\pi^2}{H^2}$. From the condition we get $\frac{\pi^3}{H^3} = \frac{2.405^2\pi^2}{2V} \Rightarrow \frac{\pi^2}{H^2} = \sqrt[3]{\frac{2.405^4\pi^4}{4V^2}}$.

Note: The radius and height of the Tarapur 3 & 4 NR in Western India is 3.192 m and 5.940 m respectively.

C2 The fuel channels are in a square arrangement (Fig-III) with nearest neighbour distance 0.286 m. The effective radius of a fuel channel (if it were solid) is 3.617×10^{-2} m. Estimate the number of fuel channels F_n in the reactor and the mass M of UO_2 required to operate the NR in steady state. [1.0]

Solution: $F_n = 387$ and $M = 9.892 \times 10^4$kg.

Detailed solution: Since the fuel channels are in square pitch of 0.286 m, the effective area per channel is 0.286^2 m^2 $= 8.180 \times 10^{-2}$ m^2.

The cross-sectional area of the core is $\pi R^2 = 3.142 \times (3.175)^2 = 31.67$ m^2, so the maximum number of fuel channels that can be accommodated in the cylinder is the integer part of $\frac{31.67}{0.0818} = 387$.

Mass of the fuel=387×Volume of the rod×density

$$= 387 \times (\pi \times 0.03617^2 \times 5.866) \times 10600 = 9.892 \times 10^4\text{kg}.$$

$F_n = 387$ [***Acceptable Range* (380 to 394)**]
$M = 9.892 \times 10^4$kg [***Acceptable Range* (9.000 to 10.00)**]

Note 1: (Not part of grading) The total volume of the fuel is $387 \times (\pi \times 0.03617^2 \times 5.866) = 9.332$ m^3. If the reactor works at 12.5 % efficieny then using the result of a-(iii) we have that the power output of the reactor is $9.332 \times 4.917 \times 10^8 \times 0.125 =$

573 MW.

Note 2: The Tarapur 3 & 4 NR in Western India has 392 channels and the mass of the fuel in it is 10.15 $\times 10^4$ kg. It produces 540 MW of power.

Alternative Solutions to sub-parts B2 and B3: Let σ be the scattering angle of the Moderator atom in the LF, taken clockwise with respect to the initial direction of the neutron before collision. Let U be the speed of the Moderator atom, in the LF, after collision. From momentum and kinetic conservation in LF we have

$$v_b = v_a \cos\theta_L + AU\cos\sigma, \quad (1)$$
$$0 = v_a \sin\theta_L - AU\sin\sigma, \quad (2)$$
$$\frac{1}{2}v_b^2 = \frac{1}{2}AU^2 + \frac{1}{2}v_a^2. \quad (3)$$

Squaring and adding eq(1) and (2) to eliminate σ and from eq(3) we get

$$A^2U^2 = v_a^2 + v_b^2 - 2v_a v_b \cos\theta_L,$$
$$A^2U^2 = Av_b^2 - Av_a^2, \quad (4)$$

which gives

$$2v_a v_b \cos\theta_L = (A+1)v_a^2 - (A-1)v_b^2. \quad (5)$$

(ii) Let v be the speed of the neutron after collision in the COMF. From definition of center of mass frame $v_m = \dfrac{v_b}{A+1}$.

$v_a \sin\theta_L$ and $v_a \cos\theta_L$ are the perpendicular and parallel components of v_a, in the LF, resolved along the initial direction of the neutron before collision. Transforming these to the COMF gives $v_a \sin\theta_L$ and $v_a\cos\theta_L - v_m$ as the perpendicular and parallel components of v. Substituting for v_m and for $2v_a v_b \cos\theta_L$ from eq(5) in $v = \sqrt{v_a^2 \sin^2\theta_L + v_a^2\cos^2\theta_L + v_m^2 - 2v_a v_m \cos\theta_L}$ and simplifying gives $v = \dfrac{Av_b}{A+1}$. Squaring the components of v to eliminate θ_L gives $v_a^2 = v^2 + v_m^2 + 2vv_m\cos\theta$. Substituting for v and v_m and simplifying gives,

$$\frac{v_a^2}{v_b^2} = \frac{E_a}{E_b} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2}.$$

$$G(\alpha,\theta) = \frac{E_a}{E_b} = \frac{A^2+1}{(A+1)^2} + \frac{2A}{(A+1)^2}\cos\theta = \frac{1}{2}\left[(1+\alpha) + (1-\alpha)\cos\theta\right].$$

(OR)

(iii) From definition of center of mass frame $v_m = \dfrac{v_b}{A+1}$. After the collision, let v and V be magnitude of the velocities of neutron and moderator atom respectively in the COMF. From conservation laws in the COMF,

$$v = AV \qquad \text{and} \qquad \frac{1}{2}(v_b - v_m)^2 + \frac{1}{2}Av_m^2 = \frac{1}{2}v^2 + \frac{1}{2}AV^2.$$

Solving gives $v = \frac{Av_b}{A+1}$ and $V = \frac{v_b}{A+1}$. We also have $v\cos\theta = v_a\cos\theta_L - v_m$, substituting for v_m and for $v_a\cos\theta_L$ from eq(5) and simplifying gives

$$\frac{v_a^2}{v_b^2} = \frac{E_a}{E_b} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2}.$$

$$G(\alpha,\theta) = \frac{E_a}{E_b} = \frac{A^2+1}{(A+1)^2} + \frac{2A}{(A+1)^2}\cos\theta = \frac{1}{2}\left[(1+\alpha)+(1-\alpha)\cos\theta\right].$$

(OR)

(iv) From definition of center of mass frame $v_m = \dfrac{v_b}{A+1}$. After the collision, let v and V be magnitude of the velocities of neutron and moderator atom respectively in the CM frame. From conservation laws in the CM frame,

$$v = AV \qquad \text{and} \qquad \frac{1}{2}(v_b - v_m)^2 + \frac{1}{2}Av_m^2 = \frac{1}{2}v^2 + \frac{1}{2}AV^2.$$

Solving gives $v = \frac{Av_b}{A+1}$ and $V = \frac{v_b}{A+1}$. $U\sin\sigma$ and $U\cos\sigma$ are the perpendicular and parallel components of U, in the LF, resolved along the initial direction of the neutron before collision. Transforming these to the COMF gives $U\sin\sigma$ and $-U\cos\sigma + v_m$ as the perpendicular and parallel components of V. So we get $U^2 = V^2\sin^2\theta + V^2\cos^2\theta + v_m^2 - 2Vv_m\cos\theta$. Since $V = v_m$ we get $U^2 = 2v_m^2(1-\cos\theta)$. Substituting for U from eq(4) and simplifying gives

$$\frac{v_a^2}{v_b^2} = \frac{E_a}{E_b} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2}.$$

$$G(\alpha,\theta) = \frac{E_a}{E_b} = \frac{A^2+1}{(A+1)^2} + \frac{2A}{(A+1)^2}\cos\theta = \frac{1}{2}\left[(1+\alpha)+(1-\alpha)\cos\theta\right].$$

Note: We have $v_a = \dfrac{\sqrt{A^2+2A\cos\theta+1}}{A+1}v_b$. Substituting for v_a, v, v_m in $v\cos\theta = v_a\cos\theta_L - v_m$ gives the relation between θ_L and θ,

$$\cos\theta_L = \frac{A\cos\theta + 1}{\sqrt{A^2+2A\cos\theta+1}}.$$

Treating the above equation as quadratic in $\cos\theta$ gives,

$$\cos\theta = \frac{-\sin^2\theta_L \pm \cos\theta_L\sqrt{A^2-\sin^2\theta_L}}{A}.$$

For $\theta_L = 0°$ the root with the negative sign gives $\theta = 180°$ which is not correct so,

$$\cos\theta = \frac{\cos\theta_L\sqrt{A^2-\sin^2\theta_L} - \sin^2\theta_L}{A}.$$

Substituting the above expression for $\cos\theta$ in the expression for $\dfrac{v_a^2}{v_b^2}$ gives an expression in terms of $\cos\theta_L$

$$\frac{v_a^2}{v_b^2} = \frac{E_a}{E_b} = \frac{A^2 + 2\cos\theta_L\sqrt{A^2-\sin^2\theta_L} + \cos 2\theta_L}{(A+1)^2}.$$