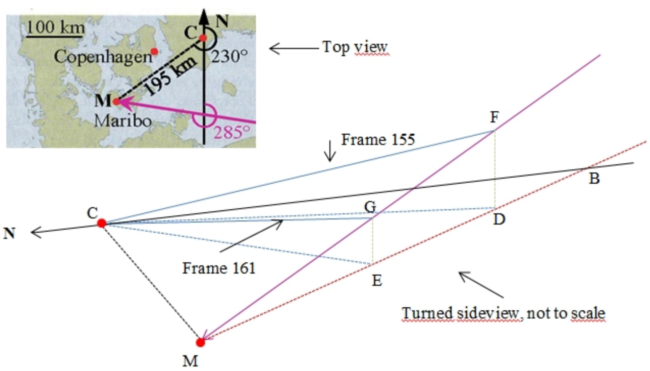
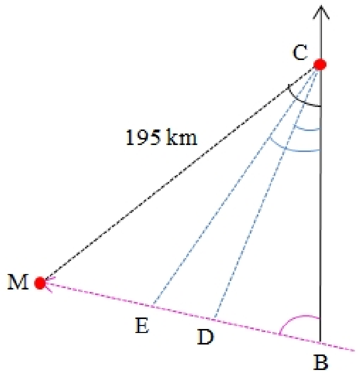
Solutions

1.1	 Top view: Triangle MCB: $\lvert CM\rvert = 195\text{ km}$, $\angle MCB = 230° - 180° = 50°$, and $\angle MBC = 75°$, so $\angle CMB = 180° - 75° - 50° = 55°$. Then $\lvert CB\rvert = \frac{\lvert CM\rvert \sin(\angle CMB)}{\sin(\angle MBC)} = 165.4\text{ km}$. Triangle DCB: $\lvert CB\rvert = 165.4\text{ km}$, $\angle DCB = 215° - 180° = 35°$, and $\angle DBC = 75°$, so $\angle CDB = 180° - 75° - 35° = 70°$. Then $\lvert CD\rvert = \frac{\lvert CB\rvert \sin(\angle DBC)}{\sin(\angle CDB)} = 170.0\text{ km}$. Triangle ECB: $\lvert CB\rvert = 165.4\text{ km}$, $\angle ECB = 221° - 180° = 41°$, and $\angle EBC = 75°$, so $\angle CEB = 180° - 75° - 41° = 64°$. Then $\lvert CE\rvert = \frac{\lvert CB\rvert \sin(\angle EBC)}{\sin(\angle CEB)} = 177.7\text{ km}$. Triangle ECD: $\angle ECD = 41° - 35° = 6°$. Horisontal distance travelled by Maribo: $\lvert DE\rvert = \frac{\lvert DC\rvert \sin(\angle ECD)}{\sin(\angle CED)} = 19.77\text{ km}$ Side view: Triangle CFD: $\lvert FD\rvert = \lvert CD\rvert \tan(\angle FCD) = 59.20\text{ km}$ Triangle CGE: $\lvert GE\rvert = \lvert CE\rvert \tan(\angle GCE) = 46.62\text{ km}$ Thus vertical distance travelled by Maribo: $\lvert FD\rvert - \lvert GE\rvert = 12.57\text{ km}$. Total distance travelled by Maribo from frame 155 to 161: $\lvert FG\rvert = \sqrt{\lvert DE\rvert^2 + (\lvert FD\rvert - \lvert GE\rvert)^2} = 23.43\text{ km}$. The speed of Maribo is $v = \frac{23.43\text{ km}}{2.28\text{ s} - 1.46\text{ s}} = 28.6\text{ km/s}$	1.2

1.2a	Newton's second law: $m_M \frac{dv}{dt} = -k\rho_{atm}\pi R_M^2 v^2$ yields $\frac{1}{v^2}dv = -\frac{k\rho_{atm}\pi R_M^2}{m_M}\,dt$. By integration $t = \frac{m_M}{k\rho_{atm}\pi R_M^2}\left(\frac{1}{0.9} - 1\right)\frac{1}{v_M} = 0.88\text{ s}$.	0.7
1.2b	$\frac{E_{kin}}{E_{melt}} = \frac{\frac{1}{2}v_M^2}{c_{sm}(T_{sm} - T_0) + L_{sm}} = \frac{4.2\times10^8}{2.1\times10^6} = 2.1\times10^2 \gg 1$.	0.3

1.3a	$[x]=[t]^{\alpha}[\rho_{sm}]^{\beta}[c_{sm}]^{\gamma}\,[k_{sm}]^{\delta}=[\mathrm{s}]^{\alpha}[\mathrm{kg\,m^{-3}}]^{\beta}[\mathrm{m^2\,s^{-2}K^{-1}}]^{\gamma}\,[\mathrm{kg\,m\,s^{-3}K^{-1}}]^{\delta}$, so $[\mathrm{m}]=[\mathrm{kg}]^{\beta+\delta}[\mathrm{m}]^{-3\beta+2\gamma+\delta}[\mathrm{s}]^{\alpha-2\gamma-3\delta}[\mathrm{K}]^{-\gamma-\delta}$. Thus $\beta+\delta=0$, $-3\beta+2\gamma+\delta=1$, $\alpha-2\gamma-3\delta=0$, and $-\gamma-\delta=0$. From which $(\alpha,\beta,\gamma,\delta)=\left(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\right)$ and $x(t)\approx\sqrt{\frac{k_{sm}t}{\rho_{sm}c_{sm}}}$.	0.6
1.3b	$x(5\text{ s})=1.6\text{ mm}$ $\quad x/R_M=1.6\text{ mm}/130\text{ mm}=0.012$.	0.4

1.4a	Rb-Sr decay scheme: ${}^{87}_{37}\mathrm{Rb}\rightarrow{}^{87}_{38}\mathrm{Sr}+{}^{0}_{-1}\mathrm{e}+\bar{\nu}_{\mathrm{e}}$	0.3
1.4b	$N_{87\mathrm{Rb}}(t)=N_{87\mathrm{Rb}}(0)\mathrm{e}^{-\lambda t}$ and Rb→Sr: $N_{87\mathrm{Sr}}(t)=N_{87\mathrm{Sr}}(0)+[N_{87\mathrm{Rb}}(0)-N_{87\mathrm{Rb}}(t)]$. Thus $N_{87\mathrm{Sr}}(t)=N_{87\mathrm{Sr}}(0)+(\mathrm{e}^{\lambda t}-1)N_{87\mathrm{Rb}}(t)$, and dividing by $N_{86\mathrm{Sr}}$ we obtain the equation of a straight line: $$\frac{N_{87\mathrm{Sr}}(t)}{N_{86\mathrm{Sr}}}=\frac{N_{87\mathrm{Sr}}(0)}{N_{86\mathrm{Sr}}}+(\mathrm{e}^{\lambda t}-1)\frac{N_{87\mathrm{Rb}}(t)}{N_{86\mathrm{Sr}}}.$$	0.7
1.4c	Slope: $\mathrm{e}^{\lambda t}-1=a=\frac{0.712-0.700}{0.25}=0.050$ and $T_{½}=\frac{\ln(2)}{\lambda}=4.9\times10^{10}$ year. So $\tau_M=\ln(1+a)\frac{1}{\lambda}=\frac{\ln(1+a)}{\ln(2)}T_{½}=3.4\times10^{9}$ year .	0.4

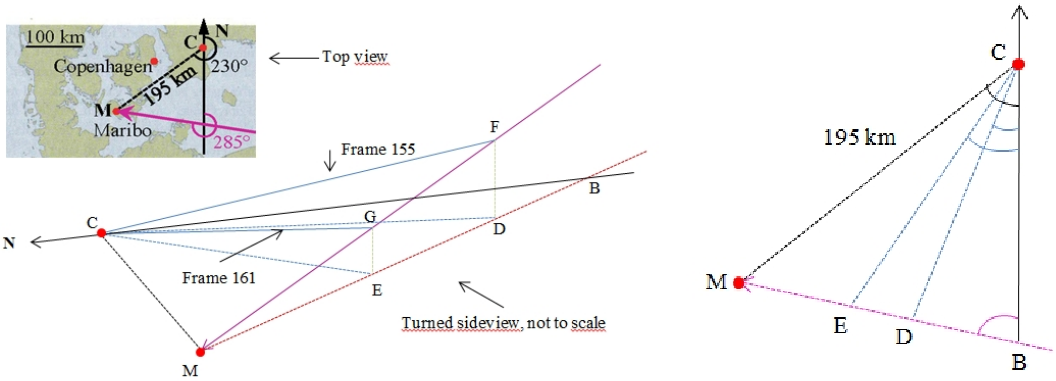
1.5	Kepler's 3rd law on comet Encke and Earth, with the orbital semi-major axis of Encke given by $a=\frac{1}{2}(a_{\min}+a_{\max})$. Thus $t_{\mathrm{Encke}}=\left(\frac{a}{a_{\mathrm{E}}}\right)^{\frac{3}{2}}t_{\mathrm{E}}=3.30\text{ year}=1.04\times10^{8}\text{ s}$.	0.6

1.6a	For Earth around its rotation axis: Angular velocity $\omega_{\mathrm{E}}=\frac{2\pi}{24\text{ h}}=7.27\times10^{-5}\text{ s}^{-1}$. Moment of inertia $I_{\mathrm{E}}=0.83\frac{2}{5}m_{\mathrm{E}}R_{\mathrm{E}}^2=8.07\times10^{37}\text{ kg m}^2$. Angular momentum $L_{\mathrm{E}}=I_{\mathrm{E}}\omega_{\mathrm{E}}=5.87\times10^{33}\text{ kg m}^2\text{s}^{-1}$. Astroid $m_{\mathrm{ast}}=\frac{4\pi}{3}R_{\mathrm{ast}}^3\rho_{\mathrm{ast}}=1.57\times10^{15}\text{ kg}$ and angular momentum $L_{\mathrm{ast}}=m_{\mathrm{ast}}v_{\mathrm{ast}}R_{\mathrm{E}}=2.51\times10^{26}\text{ kg m}^2\text{s}^{-1}$. L_{ast} is perpendicular to L_{E}, so by conservation angular momentum: $\tan(\Delta\theta)=L_{\mathrm{ast}}/L_{\mathrm{E}}=4.27\times10^{-8}$. The axis tilt $\Delta\theta=4.27\times10^{-8}$ rad (so the north pole move $R_{\mathrm{E}}\,\Delta\theta=0.27$ m).	0.7
1.6b	At vertical impact $\Delta L_{\mathrm{E}}=0$ so $\Delta(I_{\mathrm{E}}\omega_{\mathrm{E}})=0$. Thus $\Delta\omega_{\mathrm{E}}=-\omega_{\mathrm{E}}(\Delta I_{\mathrm{E}})/I_{\mathrm{E}}$, and since $\Delta I_{\mathrm{E}}/I_{\mathrm{E}}=m_{\mathrm{ast}}R_{\mathrm{E}}^2/I_{\mathrm{E}}=7.9\times10^{-10}$ we obtain $\Delta\omega_{\mathrm{E}}=-5.76\times10^{-14}\text{ s}^{-1}$. The change in rotation period is $\Delta T_{\mathrm{E}}=2\pi\left(\frac{1}{\omega_{\mathrm{E}}+\Delta\omega_{\mathrm{E}}}-\frac{1}{\omega_{\mathrm{E}}}\right)\approx-2\pi\frac{\Delta\omega_{\mathrm{E}}}{\omega_{\mathrm{E}}^2}=6.84\times10^{-5}$ s.	0.7
1.6c	At tangential impact L_{ast} is parallel to L_{E} so $L_{\mathrm{E}}+L_{\mathrm{ast}}=(I_{\mathrm{E}}+\Delta I_{\mathrm{E}})(\omega_{\mathrm{E}}+\Delta\omega_{\mathrm{E}})$ and thus $\Delta T_{\mathrm{E}}=2\pi\left(\frac{1}{\omega_{\mathrm{E}}+\Delta\omega_{\mathrm{E}}}-\frac{1}{\omega_{\mathrm{E}}}\right)=2\pi\left(\frac{I_{\mathrm{E}}+\Delta I_{\mathrm{E}}}{L_{\mathrm{E}}+L_{\mathrm{ast}}}-\frac{1}{\omega_{\mathrm{E}}}\right)=-3.62\times10^{-3}$ s.	0.7

1.7a	Minimum impact speed is the escape velocity from Earth: $v_{\mathrm{imp}}^{\mathrm{min}} = \sqrt{\frac{2Gm_E}{R_E}} = 11.2\ \mathrm{km/s}$	0.5
1.7b	Maximum impact speed $v_{\mathrm{imp}}^{\mathrm{max}}$ arises from three contributions: (I) The velocity v_{b} of the body at distance a_{E} (Earth orbit radius) from the Sun, $v_{\mathrm{b}} = \sqrt{\frac{2Gm_S}{a_E}} = 42.1\ \mathrm{km/s}$. (II) The orbital velocity of the Earth, $v_{\mathrm{E}} = \frac{2\pi a_E}{1\ \mathrm{year}} = 29.8\ \mathrm{km/s}$. (III) Gravitational attraction from the Earth and kinetic energy seen from the Earth: $\frac{1}{2}(v_{\mathrm{b}} + v_{\mathrm{E}})^2 = -\frac{Gm_E}{R_E} + \frac{1}{2}\left(v_{\mathrm{imp}}^{\mathrm{max}}\right)^2$. In conclusion: $v_{\mathrm{imp}}^{\mathrm{max}} = \sqrt{(v_{\mathrm{b}} + v_{\mathrm{E}})^2 + \frac{2Gm_E}{R_E}} = 72.8\ \mathrm{km/s}$.	1.2

	Total	**9.0**

Marking scheme

1.1	 Top view: Triangle MCB: $\lvert CM\rvert = 195$ km, $\angle MCB = 230° - 180° = 50°$, and $\angle MBC = 75°$, so $\angle CMB = 180° - 75° - 50° = 55°$. Then $\lvert CB\rvert = \frac{\lvert CM\rvert \sin(\angle CMB)}{\sin(\angle MBC)} = 165.4$ km. Triangle DCB: $\lvert CB\rvert = 165.4$ km, $\angle DCB = 215° - 180° = 35°$,and $\angle DBC = 75°$, so $\angle CDB = 180° - 75° - 35° = 70°$. Then $\lvert CD\rvert = \frac{\lvert CB\rvert \sin(\angle DBC)}{\sin(\angle CDB)} = 170.0$ km. Triangle ECB: $\lvert CB\rvert = 165.4$ km, $\angle ECB = 221° - 180° = 41°$, and $\angle EBC = 75°$, so $\angle CEB = 180° - 75° - 41° = 64°$. Then $\lvert CE\rvert = \frac{\lvert CB\rvert \sin(\angle EBC)}{\sin(\angle CEB)} = 177.7$ km. Triangle ECD:$\angle ECD = 41° - 35° = 6°$. Horisontal distance travelled by Maribo:$\lvert DE\rvert = \frac{\lvert DC\rvert \sin(\angle ECD)}{\sin(\angle CED)} = 19.77$ km Side view: Triangle CFD: $\lvert FD\rvert = \lvert CD\rvert \tan(\angle FCD) = 59.20$ km Triangle CGE: $\lvert GE\rvert = \lvert CE\rvert \tan(\angle GCE) = 46.62$ km Thus vertical distance travelled by Maribo: $\lvert FD\rvert - \lvert GE\rvert = 12.57$ km. Total distance travelled by Maribo from frame 155 to 161: $\lvert FG\rvert = \sqrt{\lvert DE\rvert^2 + (\lvert FD\rvert - \lvert GE\rvert)^2} = 23.43$ km. The speed of Maribo is $v = \frac{23.43\text{ km}}{2.28\text{ s} - 1.46\text{ s}} = 28.6$ km/s ***Marking:*** ***State*** $v = \frac{\Delta s}{\Delta t}$ ***: 0.2 points*** ***Correct drawing of all three triangles +0.4 points,*** ***Calculation of*** $\lvert DE\rvert$ ***+0.4 point*** ***Calculation of*** $\lvert FD\rvert - \lvert GE\rvert$ ***and speed +0.3 points.*** ***(-0.2 points for trivial trigonometric errors*** ***-0.1 points for more than 0.5 km/s deviation from correct result)***	1.3

1.2a	Newton's second law: $m_\mathrm{M}\frac{\mathrm{d}v}{\mathrm{d}t} = -k\rho_\mathrm{atm}\pi R_\mathrm{M}^2 v^2$ yields $\frac{1}{v^2}\mathrm{d}v = -\frac{k\rho_\mathrm{atm}\pi R_\mathrm{M}^2}{m_\mathrm{M}}\,\mathrm{d}t$. Solving by integration $t = \frac{m_\mathrm{M}}{k\rho_\mathrm{atm}\pi R_\mathrm{M}^2}\left(\frac{1}{0.9}-1\right)\frac{1}{v_\mathrm{M}} = 0.90$ s. ***Marking:*** Newton's second law: $m_\mathrm{M}\frac{\mathrm{d}v}{\mathrm{d}t} = -k\rho_\mathrm{atm}\pi R_\mathrm{M}^2 v^2$ ***(0.2 points)*** Solving by integration $t = \frac{m_\mathrm{M}}{k\rho_\mathrm{atm}\pi R_\mathrm{M}^2}\left(\frac{1}{0.9}-1\right)\frac{1}{v_\mathrm{M}} = 0.90$ s. ***(+0.3 points for correct equation for t, +0.2 points for correct value)*** Alternative solution: ***Marking:*** Newton's second law: $m_\mathrm{M}\frac{\mathrm{d}v}{\mathrm{d}t} = -k\rho_\mathrm{atm}\pi R_\mathrm{M}^2 v^2$ ***(0.2 points)*** Approximately constant acceleration from 1.46 s to 2.28 s gives $t = 0.79$ s. ***(+0.3 points for correct equation for t, +0.2 points for correct value)***	0.7
1.2b	$\frac{E_\mathrm{kin}}{E_\mathrm{melt}} = \frac{\frac{1}{2}v_\mathrm{M}^2}{c_\mathrm{sm}(T_\mathrm{sm}-T_0)+L_\mathrm{sm}} = \frac{4.2\times10^8}{2.1\times10^6} = 2.1\times10^2 \gg 1.$ ***Marking: 0.1 point for each energy term;*** ***(-0.1 if L_{sm} dropped without argument)***	0.3

1.3a	$[x] = [t]^\alpha[\rho_\mathrm{sm}]^\beta[c_\mathrm{sm}]^\gamma\,[k_\mathrm{sm}]^\delta = [\mathrm{s}]^\alpha[\mathrm{kg\,m^{-3}}]^\beta[\mathrm{m^2\,s^{-2}K^{-1}}]^\gamma\,[\mathrm{kg\,m\,s^{-3}K^{-1}}]^\delta$, so $[\mathrm{m}] = [\mathrm{kg}]^{\beta+\delta}[\mathrm{m}]^{-3\beta+2\gamma+\delta}[\mathrm{s}]^{\alpha-2\gamma-3\delta}[\mathrm{K}]^{-\gamma-\delta}$. Thus $\beta+\delta=0$, $-3\beta+2\gamma+\delta=1$, $\alpha-2\gamma-3\delta=0$, and $-\gamma-\delta=0$. From which $(\alpha,\beta,\gamma,\delta) = \left(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\right)$ and $x(t) \approx \sqrt{\frac{k_\mathrm{sm}t}{\rho_\mathrm{sm}c_\mathrm{sm}}}$. ***Marking: 0.1 points for each dimensional equation*** ***+0.2 points for solving equations correctly.***	0.6
1.3b	$x(5\text{ s}) = 1.6$ mm $\quad x/R_\mathrm{M} = 1.6\text{ mm}/130\text{ mm} = 0.012.$ ***Marking: Calculation of x: 0.3 points, calculation of x/R_M +0.1 points.***	0.4

1.4a	Rb-Sr decay scheme: ${}^{87}_{37}\mathrm{Rb} \rightarrow {}^{87}_{38}\mathrm{Sr} + {}^{0}_{-1}\mathrm{e} + \bar{\nu}_\mathrm{e}$ ***Marking: Correct scheme with both electron and antineutrino (neutrino also accepted) 0.3 points.*** ***(Missing electron or other error concerning electron: no points awarded for this question. Missing neutrino: -0.1 points)***	0.3
1.4b	$N_{87\mathrm{Rb}}(t) = N_{87\mathrm{Rb}}(0)\mathrm{e}^{-\lambda t}$ and Rb→Sr: $N_{87\mathrm{Sr}}(t) = N_{87\mathrm{Sr}}(0) + [N_{87\mathrm{Rb}}(0) - N_{87\mathrm{Rb}}(t)]$ Thus $N_{87\mathrm{Sr}}(t) = N_{87\mathrm{Sr}}(0) + (\mathrm{e}^{\lambda t} - 1)N_{87\mathrm{Rb}}(t)$, and dividing by $N_{86\mathrm{Sr}}$ we obtain $\frac{N_{87\mathrm{Sr}}(t)}{N_{86\mathrm{Sr}}} = \frac{N_{87\mathrm{Sr}}(0)}{N_{86\mathrm{Sr}}} + (\mathrm{e}^{\lambda t} - 1)\frac{N_{87\mathrm{Rb}}(t)}{N_{86\mathrm{Sr}}}$ ***Marking:*** $N_{87\mathrm{Rb}}(t) = N_{87\mathrm{Rb}}(0)\mathrm{e}^{-\lambda t}$ ***(0.2 points)*** Rb→Sr: $N_{87\mathrm{Sr}}(t) = N_{87\mathrm{Sr}}(0) + [N_{87\mathrm{Rb}}(0) - N_{87\mathrm{Rb}}(t)]$ ***(+0.2 points)*** Obtain $\frac{N_{87\mathrm{Sr}}(t)}{N_{86\mathrm{Sr}}} = \frac{N_{87\mathrm{Sr}}(0)}{N_{86\mathrm{Sr}}} + (\mathrm{e}^{\lambda t} - 1)\frac{N_{87\mathrm{Rb}}(t)}{N_{86\mathrm{Sr}}}$ ***(+0.3 points)*** ***(Wrong slope: -0.2 points, wrong intersection: -0.1)***	0.7
1.4c	Slope: $\mathrm{e}^{\lambda t} - 1 = a = \frac{0.712-0.700}{0.25} = 0.050$ and $T_{½} = \frac{\ln(2)}{\lambda} = 4.9 \times 10^{10}$ year. So $\tau_M = \ln(1+a)\frac{1}{\lambda} = \frac{\ln(1+a)}{\ln(2)}T_{½} = 3.4 \times 10^9$ year . ***Marking:*** ***Slope*** $(3.4 \pm \mathbf{0.1} \times 10^9$ year***) 0.1 point, calculation of*** τ_M ***+0.3 points.***	0.4

1.5	Kepler's 3rd law on comet Encke and Earth, with the orbital semi-major axis of Encke given by $a = \frac{1}{2}(a_\mathrm{min} + a_\mathrm{max})$. Thus $t_\mathrm{Encke} = \left(\frac{a}{a_\mathrm{E}}\right)^{\frac{3}{2}} t_\mathrm{E} = 3.30$ year $= 1.04 \times 10^8$ s. ***Marking: Correct expression for a: 0.2 points, apply Keplers third law correctly: +0.2 points, calculation of orbital period +0.2 points.***	0.6

1.6a	For Earth around its rotation axis: Angular velocity $\omega_\mathrm{E} = \frac{2\pi}{24\ \mathrm{h}} = 7.27 \times 10^{-5}\ \mathrm{s}^{-1}$. Moment of inertia $I_\mathrm{E} = 0.83\frac{2}{5}m_\mathrm{E}R_\mathrm{E}^2 = 8.07 \times 10^{37}\ \mathrm{kg\ m}^2$. Angular momentum $L_\mathrm{E} = I_\mathrm{E}\omega_\mathrm{E} = 5.87 \times 10^{33}\ \mathrm{kg\ m^2 s^{-1}}$. Astroid $m_\mathrm{ast} = \frac{4\pi}{3}R_\mathrm{ast}^3\rho_\mathrm{ast} = 1.57 \times 10^{15}$ kg and angular momentum $L_\mathrm{ast} = m_\mathrm{ast}v_\mathrm{ast}R_\mathrm{E} = 2.51 \times 10^{26}\ \mathrm{kg\ m^2 s^{-1}}$. L_ast is perpendicular to L_E, so by conservation angular momentum: $\tan(\Delta\theta) = L_\mathrm{ast}/L_\mathrm{E} = 4.27 \times 10^{-8}$. The axis tilt $\Delta\theta = 4.27 \times 10^{-8}$ rad (so the north pole move $R_\mathrm{E}\,\Delta\theta = 0.27$ m). ***Marking: Conservation of angular momentum with*** $L_\mathrm{E} = I_\mathrm{E}\omega_\mathrm{E}$ ***and*** $L_\mathrm{ast} = m_\mathrm{ast}v_\mathrm{ast}$***:0.3 point,. Realize geometry: +0.2 points, Calculation: +0.2 points.***	0.7

1.6b	At vertical impact $\Delta L_{\mathrm{E}} = 0$ so $\Delta(I_{\mathrm{E}}\omega_{\mathrm{E}}) = 0$. Thus $\Delta\omega_{\mathrm{E}} = -\omega_{\mathrm{E}}(\Delta I_{\mathrm{E}})/I_{\mathrm{E}}$, and since $\Delta I_{\mathrm{E}}/I_{\mathrm{E}} = m_{\mathrm{ast}}R_{\mathrm{E}}^2/I_{\mathrm{E}} = 7.9\times10^{-10}$ we obtain $\Delta\omega_{\mathrm{E}} = -5.76\times10^{-14}\ \mathrm{s}^{-1}$. The change in rotation period is $\Delta T_{\mathrm{E}} = 2\pi\left(\frac{1}{\omega_{\mathrm{E}}+\Delta\omega_{\mathrm{E}}} - \frac{1}{\omega_{\mathrm{E}}}\right) \approx -2\pi\frac{\Delta\omega_{\mathrm{E}}}{\omega_{\mathrm{E}}^2} = 6.84\times10^{-5}$ s. ***Marking: Conservation of angular momentum*** $\Delta(I_{\mathrm{E}}\omega_{\mathrm{E}}) = 0$: ***0.4 points, expression for*** ΔT_{E}***: +0.2 points, calculation: +0.1 points.*** ***(Low precision: -0.1 points, wrong*** $I_E + I_{ast}$***: -0.2 points)***	0.7
1.6c	At tangential impact L_{ast} is parallel to L_{E} so $L_{\mathrm{E}} + L_{\mathrm{ast}} = (I_{\mathrm{E}} + \Delta I_{\mathrm{E}})(\omega_{\mathrm{E}} + \Delta\omega_{\mathrm{E}})$ and thus $\Delta T_{\mathrm{E}} = 2\pi\left(\frac{1}{\omega_{\mathrm{E}}+\Delta\omega_{\mathrm{E}}} - \frac{1}{\omega_{\mathrm{E}}}\right) = 2\pi\left(\frac{I_{\mathrm{E}}+\Delta I_{\mathrm{E}}}{L_{\mathrm{E}}+L_{\mathrm{ast}}} - \frac{1}{\omega_{\mathrm{E}}}\right) = -3.62\times10^{-3}$ s. ***Marking: Conservation of angular momentum*** $L_{\mathrm{E}} + L_{\mathrm{ast}} = (I_{\mathrm{E}} + \Delta I_{\mathrm{E}})(\omega_{\mathrm{E}} + \Delta\omega_{\mathrm{E}})$ ***0.4 points, solving for*** ΔT_{E} ***and calculating value: 0.3 points.*** ***[Both impact directions will be accepted]*** ***(Low precision: -0.1 points).***	0.7

1.7	Maximum impact speed $v_{\mathrm{imp}}^{\mathrm{max}}$ arises from three contributions: (I) The velocity v_{b} of the body at distance a_{E} (Earth orbit radius) from the Sun, $v_{\mathrm{b}} = \sqrt{\frac{2Gm_S}{a_E}} = 42.1\ \mathrm{km/s}$. (II) The orbital velocity of the Earth, $v_{\mathrm{E}} = \frac{2\pi a_E}{1\ \mathrm{year}} = 29.8\ \mathrm{km/s}$. (III) Gravitational attraction from the Earth and kinetic energy seen from the Earth: $\frac{1}{2}(v_{\mathrm{b}} + v_{\mathrm{E}})^2 = -\frac{Gm_E}{R_E} + \frac{1}{2}\left(v_{\mathrm{imp}}^{\mathrm{max}}\right)^2$. In conclusion: $v_{\mathrm{imp}}^{\mathrm{max}} = \sqrt{(v_{\mathrm{b}} + v_{\mathrm{E}})^2 + \frac{2Gm_S}{a_E}} = 72.8\ \mathrm{km/s}$. ***Marking:*** $v_{\mathrm{b}} = \sqrt{\frac{2Gm_S}{a_E}} = 42.1\ \mathrm{km/s}$***: 0.6 points,*** $v_{\mathrm{E}} = \frac{2\pi a_E}{1\ \mathrm{year}} = 29.8\ \mathrm{km/s}$ ***: 0.4 points,*** $\frac{1}{2}(v_{\mathrm{b}} + v_{\mathrm{E}})^2 = -\frac{Gm_E}{R_E} + \frac{1}{2}\left(v_{\mathrm{imp}}^{\mathrm{max}}\right)^2$***: 0.6 points.*** ***(Alternatively using energy conservation of asteroid with static Sun and Earth, to find speed of asteroid at*** a_E ***and then adding*** v_E ***: 1.5 points).***	1.6

	Total	**9.0**

Solutions

2.1	Volume: $V = \frac{4}{3}\pi R^3 = 4.19 \times 10^{-24}\ \text{m}^3$. Mass of ions: $M = V\,\rho_{\text{Ag}} = 4.39 \times 10^{-20}$ kg no. of ions: $N = N_A \frac{M}{M_{\text{Ag}}} = 2.45 \times 10^5$. Charge density $\rho = \frac{eN}{V} = 9.38 \times 10^9\ \text{C m}^{-3}$ Electron concentration $n = \frac{N}{V} = 5.85 \times 10^{28}\ \text{m}^{-3}$, charge $Q = eN = 3.93 \times 10^{-14}$ C, and mass $m_0 = m_e N = 2.23 \times 10^{-25}$ kg.	0.7

2.2	For a sphere with radius R and constant charge density, Gauss's law yields directly $4\pi r^2 \varepsilon_0 \boldsymbol{E}_+ = \frac{4}{3}\pi r^3 \rho\, \boldsymbol{e}_r$, for $r < R$ with $\boldsymbol{e}_r$ being the unit radial vector pointing away from the center of the sphere. Thus, $\boldsymbol{E}_+ = \frac{\rho}{3\varepsilon_0}\boldsymbol{r}$. Likewise, inside a small sphere of radius R_1 and charge density $-\rho$ centered at $\boldsymbol{x}_\text{p}$ the field is $\boldsymbol{E}_- = \frac{-\rho}{3\varepsilon_0}(\boldsymbol{r} - \boldsymbol{x}_\text{p})$. Adding the two charge configurations gives the setup we want and inside the charge-free region $\lvert \boldsymbol{r} - \boldsymbol{x}_\text{p} \rvert < R_1$ the field is $\boldsymbol{E} = \boldsymbol{E}_+ + \boldsymbol{E}_- = \frac{\rho}{3\varepsilon_0}\boldsymbol{x}_\text{p}$ with the pre-factor $A = \frac{1}{3}$.	1.0

2.3	With $\boldsymbol{x}_\text{p} = x_\text{p}\,\boldsymbol{e}_x$ and $x_\text{p} \ll R$ we have from above $\boldsymbol{E}_\text{ind} = \frac{\rho}{3\varepsilon_0} x_\text{p}\,\boldsymbol{e}_x$ inside the particle. The number of electrons that produced $\boldsymbol{E}_\text{ind}$ is negligibly smaller than the number of electrons inside the particle, so $\boldsymbol{F} = Q\boldsymbol{E}_\text{ind} = (-eN)\frac{\rho}{3\varepsilon_0} x_\text{p}\,\boldsymbol{e}_x = -\frac{4\pi}{9\varepsilon_0} R^3 e^2 n^2\, \boldsymbol{x}_p$. The work done by $\boldsymbol{F}_\text{ext}$ on the electrons is $W_\text{el} = -\int_0^{x_\text{p}} F(x')\,\text{d}x' = \frac{1}{2}\left(\frac{4\pi}{9\varepsilon_0} R^3 e^2 n^2\right) x_\text{p}^2$.	0.9

2.4	The E-field is zero inside the particle, so $\boldsymbol{E}_0 + \boldsymbol{E}_\text{ind} = 0$, so $x_\text{p} = \frac{3\varepsilon_0}{\rho} E_0 = \frac{3\varepsilon_0}{en} E_0$. Charge displaced through the yz-plane: $-\Delta Q = -\rho\,\pi R^2 x_\text{p} = -\pi R^2\, ne\, x_\text{p}$.	0.5

2.5a	The electric energy W_el of a capacitor with capacitance C holding a charge $\pm\Delta Q$ is $W_\text{el} = \frac{\Delta Q^2}{2C}$, so from (3c) and (4b) we obtain $C = \frac{9}{4}\varepsilon_0 \pi R = 6.26 \times 10^{-19}$ F.	0.5
2.5b	Combining (4b) and (5a) leads $V_0 = \frac{\Delta Q}{C} = \frac{4}{3} R\, E_0$.	0.3

2.6a	$W_\text{kin} = \frac{1}{2} m_e v^2 N = \frac{1}{2} m_e v^2 \left(\frac{4}{3}\pi R^3\, n\right)$. The area πR^2 leads to $I = -e\, nv\, \pi R^2$.	0.5
2.6b	From (6a) and $W_\text{kin} = \frac{1}{2} L\, I^2$ we obtain $L = \frac{4\, m_e}{3\pi R n e^2} = 2.57 \times 10^{-14}$ H.	0.5

2.7a	$\omega_\text{p}^2 = (LC)^{-1} = ne^2(3\varepsilon_0 m_e)^{-1}$.	0.5
2.7b	$\omega_\text{p} = 7.88 \times 10^{15}$ rad/s, $\lambda_\text{p} = 2\pi c/\omega_\text{p} = 239$ nm.	0.3

2.8a	$\langle P_{\text{heat}}\rangle = \frac{1}{\tau}W_{\text{kin}} = \frac{1}{2\tau}m_e\langle v^2\rangle\left(\frac{4}{3}\pi R^3\, n\right)$. $\langle I^2\rangle = (en\,\pi R^2)^2\,\langle v^2\rangle = \left(\frac{3Q}{4R}\right)^2\langle v^2\rangle$.	1.0
2.8b	$R_{\text{heat}} = \frac{\langle P_{\text{heat}}\rangle}{\langle I^2\rangle}$ leads to $R_{\text{heat}} = \frac{W_{\text{kin}}}{\tau I^2} = \frac{2m_e}{3\pi n e^2 R\tau} = 2.46\ \Omega$.	0.8

2.9a	$[P_{\text{scat}}] = \text{J s}^{-1}$, $[Q\,x_0] = \text{C m}$, $[\omega_{\text{p}}] = \text{s}^{-1}$, $[c] = \text{m s}^{-1}$, $[\varepsilon_0] = \text{C}^2(\text{J m})^{-1}$. Note that $\left[\frac{Q^2x_0^2}{\varepsilon_0}\right] = \text{J m}^3$ so $\left[\frac{Q^2x_0^2}{\varepsilon_0}\frac{\omega_{\text{p}}^4}{c^3}\right] = \text{J s}^{-1}$. Thus $(\alpha,\beta,\gamma,\delta) = (2,4,\text{-}3,\text{-}1)$ and $P_{\text{scat}} = \frac{Q^2x_0^2\omega_{\text{p}}^4}{12\pi\varepsilon_0c^3}$.	0.7
2.9b	$R_{\text{scat}} = \frac{\langle P_{\text{scat}}\rangle}{\langle I^2\rangle}$ and $\langle v^2\rangle = \frac{1}{2}\omega_{\text{p}}^2x_0^2$ yields $R_{\text{scat}} = \frac{Q^2x_0^2\omega_{\text{p}}^4}{12\pi\varepsilon_0c^3}\frac{16R^2}{9Q^2\langle v^2\rangle} = \frac{8\omega_{\text{p}}^2R^2}{27\pi\varepsilon_0c^3} = 2.45\ \Omega$.	0.9

2.10a	At resonance the reactance of an LCR-circuit is zero: $Z_L + Z_C = 0$. The voltage-current relation then becomes: $\langle V^2\rangle = Z_R^2\langle I^2\rangle = (R_{\text{heat}} + R_{\text{scat}})^2\langle I^2\rangle$. From (2.5b) we have $\langle V^2\rangle = \frac{1}{2}V_0^2 = \frac{8}{9}R^2E_0^2$, such that $\langle I^2\rangle = \frac{8R^2E_0^2}{9(R_{\text{heat}}+R_{\text{scat}})^2}$. This leads to $\langle P_{\text{heat}}\rangle = R_{\text{heat}}\langle I^2\rangle = \frac{8R_{\text{heat}}R^2}{9(R_{\text{heat}}+R_{\text{scat}})^2}E_0^2$ and $\langle P_{\text{scat}}\rangle = \frac{R_{\text{scat}}}{R_{\text{heat}}}\langle P_{\text{heat}}\rangle$.	1.2
2.10b	$E_0 = \sqrt{2S/(\varepsilon_0c)} = 27.4\ \text{kV/m}$. $\langle P_{\text{heat}}\rangle = 6.82\ \text{nW}$. $\langle P_{\text{scat}}\rangle = 6.81\ \text{nW}$.	0.3

2.11	Time per oscillation $t_{\text{p}} = 2\pi/\omega_{\text{p}}$. Emitted energy per time $= \langle P_{\text{scat}}\rangle$. Energy per photon $E_{\text{p}} = \hbar\omega_{\text{p}}$. Numbers of oscillation per photon $N_{\text{osc}} = \frac{\hbar\omega_{\text{p}}}{t_{\text{p}}\langle P_{\text{scat}}\rangle} = \frac{\hbar\omega_{\text{p}}^2}{2\pi\langle P_{\text{scat}}\rangle} = 1.53\times10^5$.	0.6

2.12a	Total number of nanoparticles: $N_{\text{np}} = h^2a\,n_{\text{np}} = 7.3\times10^{11}$. Total power into steam from Joule heating: $P_{\text{st}} = N_{\text{np}}P_{\text{heat}} = 4.98\ \text{kW}$. In terms of heating up a mass of steam per second μ_{st}: $P_{\text{st}} = \mu_{\text{st}}L_{\text{tot}}$, with $L_{\text{tot}} = c_{\text{wa}}(T_{100} - T_{\text{wa}}) + L_{\text{wa}} + c_{\text{st}}(T_{\text{st}} - T_{100})$ $= 2.62\times10^6\ \text{J kg}^{-1}$. Thus $\mu_{\text{st}} = \frac{P_{\text{st}}}{L_{\text{tot}}} = 1.90\times10^{-3}\ \text{kg s}^{-1}$.	0.6
2.12b	$P_{\text{tot}} = h^2S = 0.01\text{m}^2 \times 1\ \text{MW m}^{-2} = 10.0\ \text{kW}$, and thus $\eta = \frac{P_{\text{st}}}{P_{\text{tot}}} = \frac{4.98\ \text{kW}}{10.0\ \text{kW}} = 0.498$.	0.2

	Total	**12.0**

Marking scheme

General issues

- Number of significant digits: 3, no punishment if 2 or 4, punishment -0.05 if 1 or >4.
- Rounding: no punishment (for example, students result 2.34, correct result 2.47): no punishment if there is small (+/-1) difference in the second from the left significant digit, provided that the formula is correct.
- Wrong sign: depends on the assignment (no punishment for charge or current, but punishment for the wrong energy: -0.1 pt)
- Cumulative mistakes:
  - we punish only one time, except for the case when the result is physically wrong (for example, efficiency > 100%) or the dimensions are wrong (m=s^2 ~30-50%),
- Calculations:
  - If the pre-factor in the formula is wrong, but the value according to this formula is calculated correct, the student receives points for the value,
  - if the formula is incorrect (dimensions), no points for the value.
- Forgotten units:
  - no punishment, if there are no units on the answer sheet, but there are units on the draft papers,
  - punishment deduct half the points for this point, if there are no units anywhere.
- Wrong fomula:
  - wrong dimensions (m=s^2): deduct 50% of the value.
  - wrong pre-factor: -0.1
- We round up the points for each section to higher value in steps of 0.1 (eg. 12.05 -> 12.1).

2.1

Volume: $V = \frac{4}{3}\pi R^3 = 4.19 \times 10^{-24}\ \mathrm{m}^3$	0.1
Mass of ions: $M = V\ D_{\mathrm{Ag}} = 4.39 \times 10^{-20}\ \mathrm{kg}$	0.1
no. of ions: $N = N_A \frac{M}{M_{\mathrm{Ag}}} = 2.45 \times 10^5$	0.1
Charge density $\rho = \frac{eN}{V} = 9.38 \times 10^9\ \mathrm{C\ m}^{-3}$	0.1
Electron concentration $n = \frac{N}{V} = 5.85 \times 10^{28}\ \mathrm{m}^{-3}$	0.1
charge $Q = eN = 3.93 \times 10^{-14}$ C, No punishment for the sign	0.1
$m_0 = m_e N = 2.23 \times 10^{-25}$ kg	0.1

- Correct 0.1,
- punishment for wrong value or no units -0.05
- round up to the higher value

2.2

Show that the electric $\boldsymbol{E} = A\,(\rho/\varepsilon_0)\,\boldsymbol{x}_\mathrm{d}$, and pre-factor $A = \frac{1}{3}$ Check derivation in the papers Electric field in side a homogeneous sphere: 0.5 (just Gauss law 0.2) Principle of superposition:0.4 Vector calculations:0.2 Correct A: 0.1	1.2

2.3

$\boldsymbol{F} = Q\boldsymbol{E}_\mathrm{ind} = (-en)\frac{\rho}{3\varepsilon_0}x_\mathrm{p}\,\boldsymbol{e}_x = -\frac{4\pi}{9\varepsilon_0}R^3e^2n^2x_\mathrm{p}\,\boldsymbol{e}_x$ F=Q Eind: 0.4 pt If F = ½ Q Eind: 0.2 pt Correct calculation: +0.1 pt	0.5
$W_\mathrm{el} = -\int_0^{x_\mathrm{p}} F(x')\,\mathrm{d}x' = \frac{1}{2}\left(\frac{4\pi}{9\varepsilon_0}R^3e^2n^2\right)x_\mathrm{p}^2$ If integral: 0.4 If maxforce times distance: 0.2 Calculation of expression +0.1	0.5

punishment for the wrong sign 0.1 only for the negative work, not force

2.4

$x_p = \frac{3\varepsilon_0 E_0}{en}$ Mentioned that electric field in equilibrium is 0: 0.1 pt	0.3
$-\Delta Q = -\rho\,\pi R^2 x_0 = -\pi R^2\,ne\,x_0$ No punishment for sign	0.3

2.5

a	$C = \frac{9}{4}\varepsilon_0\pi R$ Formula for the energy of the capacitor W=q^2/2C: 0.2 Try to estimate as a flat capacitor: 0.3 Just applying the formula for the capacitance of a single sphere: 0 Dimensional analysis: 0	0.6
a	6.26×10^{-19} F	0.1
b	$V_0 = \frac{\Delta Q}{C} = \frac{4}{3}R\,E_0$ Estimation of the voltage as 2ER: 0.2 pt	0.4

2.6

a	$W_{\text{kin}} = \frac{1}{2} m_e v^2 N = \frac{1}{2} m_e v^2 \left(\frac{4}{3}\pi R^3\ n\right)$ Punishment for forgetting N: -0.2 pt	0.4
a	$I = -e\ nv\ \pi R^2$ No punishment for the sign	0.3
b	$L = \frac{4\ m_e}{3\pi R n e^2}$ Energy of the inductor: 0.2 Dimensional analysis: 0	0.4
b	2.57×10^{-14} H	0.1

2.7

a	$\omega_{\text{p}}^2 = (LC)^{-1} = ne^2 (3\varepsilon_0 m_e)^{-1}$ LC-circuit frequency: 0.3 Dimensional analysis 0.2 Alternative ways	0.5
b	$\omega_{\text{p}} = 7.88 \times 10^{15}$ rad/s, No punishment for Hz	0.1
b	$\lambda_{\text{p}} = 2\pi c/\omega_{\text{p}} = 239$ nm Correct formula 0.2 No punishment for m	0.3

2.8

a	$\langle P_{\text{heat}} \rangle = \frac{1}{\tau} W_{\text{kin}} = \frac{1}{2\tau} m_e \langle v^2 \rangle \left(\frac{4}{3}\pi R^3\ n\right)$ $\langle P_{heat} \rangle = \frac{\langle W_{kin} \rangle}{\tau} = \frac{1}{2\tau} m_e$ .... Formula for power as mean kinetic energy divided over tau: 0.4 If calculated in the assumption of constant acceleration: -0.2	0.5
a	$\langle I^2 \rangle = (en\ \pi R^2)^2\ \langle v^2 \rangle = \left(\frac{3Q}{4R}\right)^2 \langle v^2 \rangle$	0.5
b	$R_{\text{heat}} = \frac{W_{\text{kin}}}{\tau I^2} = \frac{2 m_e}{3\pi n e^2 R \tau}$ Formula P=R<I^2>: 0.3	0.8
b	$2.46\ \Omega$	0.2

2.9

$R_{scat} = \frac{Q^2 x_0^2 \omega_p^4}{12\pi\varepsilon_0 c^3} \frac{16R^2}{9Q^2\langle v^2\rangle} = \frac{8\omega_0^2 R^2}{27\pi\varepsilon_0 c^3}$ If they made <v^2> correct 0.4 pt If <v^2>=x0^2 omega^2: -0.2 P=I^2Rsc gives 0.3 pt	0.8
$2.45\ \Omega$	0.2

2.10

a	$\langle P_{heat}\rangle = R_{heat}\langle I^2\rangle = \frac{8R_{heat}R^2}{9(R_{heat}+R_{scat})^2} E_0^2$	0.9 (0.3)
a	$\langle P_{scat}\rangle$ If at least one correct 0.9 If bulk expression with L and C: deduce 0.1 pt Vector diagram or differential equation: 0.3 Drawing equivalent RLC circuit: 0.2 If the student connect resistances in parallel: 0	0.3 (0.9)
b	$E_0 = \sqrt{2S/(\varepsilon_0 c)} = 27.4\ \mathrm{kV/m}$	0.1
b	$\langle P_{heat}\rangle = 6.82\ \mathrm{nW}$	0.1
b	$\langle P_{scat}\rangle = 6.81\ \mathrm{nW}$	0.1

2.11

$\mu_{st} \quad \frac{P_{st}}{L_{tot}} = 1.90 \times 10^{-3}\ \mathrm{kg\ s^{-1}}$ Correct formula 0.4	0.6
$\eta = \frac{P_{st}}{P_{tot}} = \frac{4.98\ \mathrm{kW}}{10.0\ \mathrm{kW}} = 0.498$ If effficiency is calculated through the powers in single particle: 0	0.2

Solutions

3.1	The pressure is given by the hydrostatic pressure $p(x,z) = \rho_{\text{ice}} g (H(x) - z)$, which is zero at the surface.	0.3

3.2a	The outward force on a vertical slice at a distance x from the middle and of a given width Δy is obtained by integrating up the pressure times the area: $$F(x) = \Delta y \int_0^{H(x)} \rho_{\text{ice}}\, g\, (H(x) - z)\, \mathrm{d}z\ = \frac{1}{2}\, \Delta y\, \rho_{\text{ice}}\, g\, H(x)^2$$ which implies that $\Delta F = F(x) - F(x + \Delta x) = -\frac{\mathrm{d}F}{\mathrm{d}x} \Delta x = -\Delta y\, \rho_{\text{ice}}\, g\, H(x) \frac{\mathrm{d}H}{\mathrm{d}x} \Delta x$. This finally shows that $$S_{\text{b}} = \frac{\Delta F}{\Delta x \Delta y} = -\, \rho_{\text{ice}}\, g\, H(x) \frac{\mathrm{d}H}{\mathrm{d}x}$$ Notice the sign, which must be like this, since S_b was defined as positive and $H(x)$ is a decreasing function of x.	0.9
3.2b	To find the height profile, we solve the differential equation for $H(x)$: $$-\frac{S_{\text{b}}}{\rho_{\text{ice}}\, g} = H(x) \frac{\mathrm{d}H}{\mathrm{d}x} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} H(x)^2$$ with the boundary condition that $H(L) = 0$. This gives the solution: $$H(x) = \sqrt{\frac{2 S_b L}{\rho_{\text{ice}}\, g}} \sqrt{1 - x/L}$$ Which gives the maximum height $H_{\text{m}} = \sqrt{\frac{2 S_b L}{\rho_{\text{ice}}\, g}}$. Alternatively, dimensional analysis could be used in the following manner. First notice that $\mathcal{L} = [H_{\text{m}}] = [\rho_{\text{ice}}^{\alpha}\, g^{\beta} \tau_{\text{b}}^{\gamma} L^{\delta}]$. Using that $[\rho_{\rho_{\text{ice}}}] = \mathcal{M}\mathcal{L}^{-3}$, $[g] = \mathcal{L}\,\mathcal{T}^{-2}$, $[\tau_b] = \mathcal{M}\mathcal{L}^{-1}\,\mathcal{T}^{-2}$, demands that $\mathcal{L} = [H_{\text{m}}] = [{\rho_i}^{\alpha} g^{\beta} {\tau_b}^{\gamma} L^{\delta}] = \mathcal{M}^{\alpha+\gamma} \mathcal{L}^{-3\alpha+\beta-\gamma+\delta}\, \mathcal{T}^{-2\beta-2\gamma}$, which again implies $\alpha + \gamma = 0$, $-3\alpha + \beta - \gamma + \delta = 1$, $2\beta + 2\gamma = 0$. These three equations are solved to give $\alpha = \beta = -\gamma = \delta - 1$, which shows that $$H_{\text{m}} \propto \left(\frac{S_{\text{b}}}{\rho_{\rho_{\text{ice}}} g}\right)^{\gamma} L^{1-\gamma}$$ Since we were informed that $H_{\text{m}} \propto \sqrt{L}$, it follows that $\gamma = 1/2$. With the boundary condition $H(L) = 0$, the solution then take the form $$H(x) \propto \left(\frac{S_{\text{b}}}{\rho_{\text{ice}}\, g}\right)^{1/2} \sqrt{L - x}$$ The proportionality constant of $\sqrt{2}$ cannot be determined in this approach.	0.8

3.2c	For the rectangular Greenland model, the area is equal to $A = 10L^2$ and the volume is found by integrating up the height profile found in problem 3.2b: $$V_{\mathrm{G,ice}} = (5L)2\int_0^L H(x)\,\mathrm{d}x = 10L\int_0^L \left(\frac{\tau_\mathrm{b} L}{\rho_\mathrm{ice}\, g}\right)^{1/2} \sqrt{1-x/L}\,\mathrm{d}x = 10H_\mathrm{m}L^2\int_0^1 \sqrt{1-\tilde{x}}\,\mathrm{d}\tilde{x}$$ $$= 10H_\mathrm{m}L^2\left[-\frac{2}{3}(1-\tilde{x})^{3/2}\right]_0^1 = \frac{20}{3}H_\mathrm{m}L^2 \propto L^{5/2},$$ where the last line follows from the fact that $H_\mathrm{m} \propto \sqrt{L}$. Note that the integral need not be carried out to find the scaling with L. This implies that $V_{\mathrm{G,ice}} \propto {A_G}^{5/4}$ and the wanted exponent is $\gamma = 5/4$.	0.5

3.3	According to the assumption of constant accumulation c the total mass accumulation rate from an area of width Δy between the ice divide at $x = 0$ and some point at $x > 0$ must equal the total mass flux through the corresponding vertical cross section at x. That is: $\rho c x \Delta y = \rho \Delta y H_\mathrm{m} v_x(x)$, from which the velocity is isolated: $$v_x(x) = \frac{cx}{H_\mathrm{m}}$$	0.6

3.4	From the given relation of incompressibility it follows that $$\frac{\mathrm{d}v_z}{\mathrm{d}z} = -\frac{\mathrm{d}v_x}{\mathrm{d}x} = -\frac{c}{H_\mathrm{m}}$$ Solving this differential equation with the initial condition $v_z(0) = 0$, shows that: $$v_z(z) = -\frac{cz}{H_\mathrm{m}}$$	0.6

3.5	Solving the two differential equations $$\frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{cz}{H_\mathrm{m}} \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{cx}{H_\mathrm{m}}$$ with the initial conditions that $z(0) = H_\mathrm{m}$, and $x(0) = x_i$ gives $$z(t) = H_\mathrm{m}\,\mathrm{e}^{-ct/H_\mathrm{m}} \quad \text{and} \quad x(t) = x_i\,\mathrm{e}^{ct/H_\mathrm{m}}$$ This shows that $z = H_\mathrm{m}\,x_i\,/x$, meaning that flow lines are hyperbolas in the xz-plane. Rather than solving the differential equations, one can also use them to show that $$\frac{\mathrm{d}}{\mathrm{d}t}(xz) = \frac{\mathrm{d}x}{\mathrm{d}t}z + x\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{cx}{H_\mathrm{m}}z - x\frac{cz}{H_\mathrm{m}} = 0$$ which again implies that $xz = \text{const}$. Fixing the constant by the initial conditions, again leads to the result that $z = H_\mathrm{m}x_i/x$.	0.9

3.6	At the ice divide, $x = 0$, the flow will be completely vertical, and the t-dependence of z found in 3.5 can be inverted to find $\tau(z)$. One finds that $\tau(z) = \frac{H_\mathrm{m}}{c}\ln\left(\frac{H_\mathrm{m}}{z}\right)$.	1.0

3.7a	The present interglacial period extends to a depth of 1492 m, corresponding to 11,700 year. Using the formula for $\tau(z)$from problem 3.6, one finds the following accumulation rate for the interglacial: $$c_{\mathrm{ig}} = \frac{H_{\mathrm{m}}}{11{,}700\ \text{years}} \ln\left(\frac{H_{\mathrm{m}}}{H_{\mathrm{m}} - 1492\ \mathrm{m}}\right) = 0.17\ \mathrm{m/year}.$$ The beginning of the ice age 120,000 years ago is identified as the drop in $\delta^{18}\mathrm{O}$ in figure 3.2b at a depth of 3040 m. Using the vertical flow velocity found in problem 3.4, on has $\frac{\mathrm{d}z}{z} = -\frac{c}{H_{\mathrm{m}}}\mathrm{d}t$, which can be integrated down to a depth of 3040 m, using a stepwise constant accumulation rate: $$H_{\mathrm{m}} \ln\left(\frac{H_{\mathrm{m}}}{H_{\mathrm{m}} - 3040\ \mathrm{m}}\right) = -H_{\mathrm{m}} \int_{H_{\mathrm{m}}}^{H_{\mathrm{m}}-3040\ \mathrm{m}} \frac{1}{z}\mathrm{d}z$$ $$= \int_{11{,}700\ \text{year}}^{120{,}000\ \text{year}} c_{\mathrm{ia}}\ \mathrm{d}t + \int_{0}^{11{,}700\ \text{year}} c_{\mathrm{ig}}\ \mathrm{d}t$$ $$= c_{\mathrm{ia}}(120{,}000\ \text{year-}11{,}700\ \text{year}) + c_{\mathrm{ig}}11{,}700\ \text{year}$$ Isolating form this equation leads to $c_{\mathrm{ia}} = 0.12$, i.e. far less precipitation than now.	0.8
3.7b	Reading off from figure 3.2b: $\delta^{18}\mathrm{O}$ changes from $-43{,}5$ ‰ to $-34{,}5$ ‰. Reading off from figure 3.2a, T then changes from -40 °C to -28 °C. This gives $\Delta T \approx 12$ °C.	0.2

3.8a	From the area A_{G} one finds that $L = \sqrt{A_{\mathrm{G}}/10} = 4.14 \times 10^{5}$ m. Inserting numbers in the volume formula found in 3.2c, one finds that: $$V_{\mathrm{G,ice}} = \frac{20}{3} L^{5/2} \sqrt{\frac{2S_{\mathrm{b}}}{\rho_{\mathrm{ice}} g}} = 3.46 \times 10^{15}\ \mathrm{m}^{3}$$ This ice volume must be converted to liquid water volume, by equating the total masses, i.e. $V_{\mathrm{G,wa}} = V_{\mathrm{G,ice}} \frac{\rho_{\mathrm{ice}}}{\rho_{\mathrm{wa}}} = 3.17 \times 10^{15}\ \mathrm{m}^{3}$, which is finally converted to a sea level rise, as $h_{\mathrm{G,rise}} = \frac{V_{\mathrm{G,wa}}}{A_{\mathrm{o}}} = 8.78$ m.	0.6
3.8b	From the area and aspect ratio of Antarctica and the volume to area scaling law found in problem 3.2c, it follows that $$\frac{h_{\mathrm{A,rise}}}{h_{\mathrm{G,rise}}} = \frac{V_{\mathrm{A,wa}}}{V_{\mathrm{G,wa}}} = \frac{2}{5}\left(\frac{L_{\mathrm{A}}}{L_{\mathrm{G}}}\right)^{5/2} = \frac{2}{5}\left(\frac{5}{2}\frac{A_{\mathrm{A}}}{A_{\mathrm{G}}}\right)^{5/4} = \left(\frac{5}{2}\right)^{1/4}\left(\frac{A_{\mathrm{A}}}{A_{\mathrm{G}}}\right)^{5/4}$$ which gives a sea level rise of $h_{\mathrm{A,rise}} = 127$ m.	0.2

3.9	From solving problem 3.8a, the total volume of the ice is known. From this number, we first deduce a radius of the spherical model of the Greenlandic ice sheet: $$R_{\mathrm{ice}} = \left(\frac{3\ V_{\mathrm{G,ice}}}{4\pi}\right)^{1/3} = \left(\frac{3 \times 3.46 \times 10^{15}}{4\pi}\right)^{1/3} \mathrm{m} = 93.8\ \mathrm{km}$$ This is an order of magnitude larger than the average sea depth of roughly 3 km, and therefore it doesn't matter much if we locate the sphere at the mean Earth radius, rather	1.6

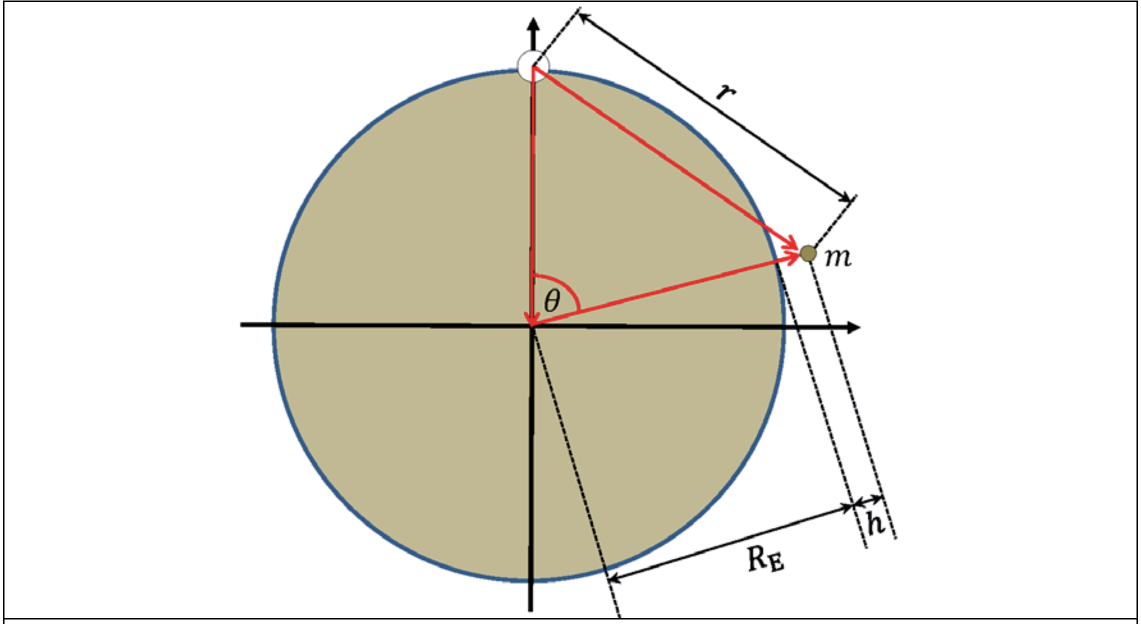


Figure 3.S1 Geometry of the ice ball (white circle) with a test mass m (small gray circle).

than sea level. The total mass of the ice is

$$M_{\text{ice}} = V_{\text{G,ice}}\,\rho_{\text{ice}} = 3.17\times10^{18}\ \text{kg} = 5.31\times10^{-7} m_{\text{E}}$$

The total gravitational potential felt by a test mass m at a certain height h above the surface of the Earth, and at a polar angle θ (cf. figure 3.S1), with respect to a rotated polar axis going straight through the ice sphere is found by adding that from the Earth with that from the ice:

$$U_{\text{tot}} = -\frac{G m_{\text{E}} m}{R_{\text{E}}+h} - \frac{G M_{\text{ice}} m}{r} = -mgR_E\left(\frac{1}{1+h/R_E} + \frac{M_{ice}/m_E}{r/R_E}\right)$$

where $g = Gm_E/R_E^2$. Since $h/R_{\text{E}} \ll 1$ one may use the approximation given in the problem, $(1+\text{x})^{-1} \approx 1-x,\ \ |x| \ll 1$, to approximate this by

$$U_{\text{tot}} \approx -mgR_E\left(1 - \frac{h}{R_E} + \frac{M_{ice}/m_E}{r/R_E}\right).$$

Isolating h now shows that $h = h_0 + \frac{M_{ice}/m_E}{r/R_E} R_E$, where $h_0 = R_E + U_{\text{tot}}/(mg)$. Using again that $h/R_{\text{E}} \ll 1$, trigonometry shows that $r \approx 2R_{\text{E}}|\sin(\theta/2)|$, and one has:

$$h(\theta) - h_0 \approx \frac{M_{\text{ice}}/m_{\text{E}}}{2|\sin(\theta/2)|} R_E \approx \frac{1.69\ \text{m}}{|\sin(\theta/2)|}.$$

To find the magnitude of the effect in Copenhagen, the distance of 3500 km along the surface is used to find the angle $\theta_{\text{CPH}} = (3.5\times10^6\ \text{m})/R_E \approx 0.55$, corresponding to $h_{\text{CPH}} - h_0 \approx 6.2$ m. Directly opposite to Greenland corresponds to $\theta = \pi$, which gives $h_{\text{OPP}} - h_0 \approx 1.7$ m. From the radius of the ice ball, one finds that $\theta_{\text{GRL}} = R_{ice}/R_E \approx 0.0147$, which corresponds to $h_{\text{GRL}} - h_0 \approx 229.8$ m. Finally, the differences become $h_{\text{GRL}} - h_{\text{CPH}} \approx 223.5$ m, and $h_{\text{CPH}} - h_{\text{OPP}} \approx 4.5$ m, where h_0 has dropped out.

	Total	**9.0**

Marking scheme

3.1	$p(x,z) = \rho_{\text{ice}} g (H(x) - z)$ Getting hydrostatic pressure right gives 0.3. Deduct 0.2 for writing $H(x)$ or z instead of $H(x) - z$ (wrong boundary condition) Deduct 0.1 for writing H_m instead of $H(x)$.	0.3

3.2a	$k = -\rho_{\text{ice}}\, g$ 0.5 points are given for solving the problem by dimensional analysis, which misses an overall pre-factor. No deduction for the sign, since one could in principle have looked at negative x.	0.9
3.2b	$H(x) = \sqrt{\frac{2S_b L}{\rho_{\text{ice}}\, g}} \sqrt{1 - x/L}$ Setting up and solving the differential equation (integrating) gives 0.5. Implementing the correct boundary conditions gives 0.3 No deductions /additions for absolute value around x. Deduct 0.1 for missing factor of $\sqrt{2}$ due to simple arithmetic error. Deduct 0.2 for solving problem only by dimensional analysis (+boundary condition), which also misses the factor of $\sqrt{2}$ and leaves the pre-factor unknown. Deduct 0.4 for applying wrong boundary condition at $x = L$. Deduct 0.2 for expressing solution in terms of H_m, but not actually finding expression for H_m in terms of the given parameters.	0.8
3.2c	$\gamma = 5/4$ Give 0.3 for getting the correct idea, involving an integral over the height profile times the ground-area along y. Give 0.2 for getting the scaling of V with L correct. Deduct 0.1 for misunderstanding the definition of γ, as in e.g. finding the scaling with L rather than with A.	0.5

3.3	$v_x(x) = \frac{cx}{H_\mathrm{m}}$ Setting up mass balance and solving the problem gives 0.6. Dimensional analysis cannot be used here since x/H_m is dimensionless. Deduct 0.2 for getting the sign and/or pre-factor wrong.	0.6

3.4	$v_z(z) = -\frac{cz}{H_\mathrm{m}}$ Setting up the correct differential equation and solve it gives 0.6. No deduction for using $v_x(x)$ without having solved for this first (i.e. without solving problem 3.3) Deduct 0.2 for getting the sign and/or pre-factor wrong. Deduct 0.2 for wrong implementation of the boundary condition, like keeping an undetermined integration constant.	0.6

3.5	$z = H_\mathrm{m} x_i / x$ Finding that $z \propto 1/x$ gives 0.6. If done by solving differential equation, then give 0.4 for setting up the equation and 0.2 for integrating it. Implementing the initial condition gives 0.3. No deduction if they happen to write $H(x)$ instead of H_m.	0.9

3.6	$\tau(z) = \frac{H_\mathrm{m}}{c} \ln\left(\frac{H_\mathrm{m}}{z}\right)$ Setting up the right equation, including the integral gives 0.4. Integrating (solving) the equation gives 0.3. Implying the correct physical boundary conditions gives 0.3.	1.0

3.7a	$c_{\text{ig}} = 0.17\ \text{m/year}$ $c_{\text{ia}} = 0.12\ \text{m/year}$ Getting the correct value for c_{ig} gives 0.3. Getting the correct value for c_{ia} gives 0.5. We will accept a range of significant digits between 2 and 4 without deduction. If c_{ia} is calculated by exactly the same method as c_{ig} (i.e. neglecting the influence of the previous interglacial age) deduct 0.3 (i.e. then the nearly correct value ($c_{\text{ia}} = 0.13\ \text{m/year}$) obtained this way will give a total of 0.2).	0.8
3.7b	$\Delta T \approx 12\ ℃$ Given that the data are very noisy, give all 0.2 points for finding a number of $\Delta T = 12\ ℃ \pm 4\ ℃$. Give only 0.1 for $\Delta T = 12\ ℃ \pm 8\ ℃$, (and outside $\Delta T = 12\ ℃ \pm 4\ ℃$), unless it is supplemented by a qualified and valid discussion, which can then allow giving 0.2. No deduction for wrong sign on ΔT. Deduct 0.1 for missing units on ΔT.	0.2

3.8	$h_{\mathrm{G,rise}} = 8.78\ \mathrm{m}$ We subdivide the problem into four basic elements which are bound to enter the solution in some way. Give 0.1 for finding the correct value of L. Give 0.2 for finding the correct values of V_{ice}. Give 0.2 for converting ice volume to water volume, using the two different given densities. Give 0.1 for converting the water volume to a sea-level rise.	0.6

3.9	$h_{\mathrm{CPH}} - h_{\mathrm{OPP}} = 4.5\ \mathrm{m}$ In grading this problem, we are searching for correct elements in three areas, and one can earn up to 0.6 points from each of the following three categories: 1) Stating the relevant physical principles (fx. Involving gravitational potentials from two bodies, sea level as equipotential surface, etc.) and putting them together in a meaningful way. 2) Writing up the correct potentials, with the relevant lengths, masses, etc., and expressing the ideas from pt.1 in correct mathematical form. 3) Actually solving the equations, making the appropriate expansions/approximations, carrying out geometry/trigonometry.	1.8

	Total	**9.0**