SOLUTIONS to Theory Question 1

Geometry Each side of the diamond has length $L = \dfrac{a}{\cos\theta}$ and the distance between parallel sides is $D = \dfrac{a}{\cos\theta}\sin(2\theta) = 2a\sin\theta$. The area is the product thereof, $A = LD$, giving

1.1

$$A = 2a^2\tan\theta\,.$$

The height H by which a tilt of ϕ lifts OUT1 above IN is $H = D\sin\phi$ or

1.2

$$H = 2a\sin\theta\,\sin\phi\,.$$

Optical path length Only the two parallel lines for IN and OUT1 matter, each having length L. With the de Broglie wavelength λ_0 on the IN side and λ_1 on the OUT1 side, we have

$$\Delta N_{\text{opt}} = \frac{L}{\lambda_0} - \frac{L}{\lambda_1} = \frac{a}{\lambda_0\cos\theta}\left(1 - \frac{\lambda_0}{\lambda_1}\right).$$

The momentum is h/λ_0 or h/λ_1, respectively, and the statement of energy conservation reads

$$\frac{1}{2M}\left(\frac{h}{\lambda_0}\right)^2 = \frac{1}{2M}\left(\frac{h}{\lambda_1}\right)^2 + MgH\,,$$

which implies

$$\frac{\lambda_0}{\lambda_1} = \sqrt{1 - 2\frac{gM^2}{h^2}\lambda_0^2 H}\,.$$

Upon recognizing that $(gM^2/h^2)\lambda_0^2 H$ is of the order of 10^{-7}, this simplifies to

$$\frac{\lambda_0}{\lambda_1} = 1 - \frac{gM^2}{h^2}\lambda_0^2 H\,,$$

and we get

$$\Delta N_{\text{opt}} = \frac{a}{\lambda_0\cos\theta}\frac{gM^2}{h^2}\lambda_0^2 H$$

or

1.3 $$\Delta N_{\text{opt}} = 2\frac{gM^2}{h^2}a^2\lambda_0 \tan\theta \, \sin\phi \, .$$

A more compact way of writing this is

1.4 $$\Delta N_{\text{opt}} = \frac{\lambda_0 A}{V} \sin\phi \, ,$$

where

1.4 $$V = 0.1597 \times 10^{-13}\,\text{m}^3 = 0.1597\,\text{nm}\,\text{cm}^2$$

is the numerical value for the volume parameter V.

There is constructive interference (high intensity in OUT1) when the optical path lengths of the two paths differ by an integer, $\Delta N_{\text{opt}} = 0, \pm 1, \pm 2, \ldots$, and we have destructive interference (low intensity in OUT1) when they differ by an integer plus half, $\Delta N_{\text{opt}} = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$. Changing ϕ from $\phi = -90°$ to $\phi = 90°$ gives

$$\Delta N_{\text{opt}}\Big|_{\phi=-90°}^{\phi=90°} = \frac{2\lambda_0 A}{V} \, ,$$

which tell us that

1.5 $$\sharp \text{ of cycles} = \frac{2\lambda_0 A}{V} \, .$$

Experimental data For $a = 3.6\,\text{cm}$ and $\theta = 22.1°$ we have $A = 10.53\,\text{cm}^2$, so that

1.6 $$\lambda_0 = \frac{19 \times 0.1597}{2 \times 10.53}\,\text{nm} = 0.1441\,\text{nm} \, .$$

And 30 full cycles for $\lambda_0 = 0.2\,\text{nm}$ correspond to an area

1.7 $$A = \frac{30 \times 0.1597}{2 \times 0.2}\,\text{cm}^2 = 11.98\,\text{cm}^2 \, .$$

SOLUTIONS to Theory Question 2

Basic relations Position $\tilde{x}$ shows up on the picture if light was emitted from there at an instant that is earlier than the instant of the picture taking by the light travel time T that is given by

$$T = \sqrt{D^2 + \tilde{x}^2}\Big/c\,.$$

During the lapse of T the respective segment of the rod has moved the distance vT, so that its actual position x at the time of the picture taking is

2.1
$$x = \tilde{x} + \beta\sqrt{D^2 + \tilde{x}^2}\,.$$

Upon solving for $\tilde{x}$ we find

2.2
$$\tilde{x} = \gamma^2 x - \beta\gamma\sqrt{D^2 + (\gamma x)^2}\,.$$

Apparent length of the rod Owing to the Lorentz contraction, the actual length of the moving rod is L/γ, so that the actual positions of the two ends of the rod are

$$x_\pm = x_0 \pm \frac{L}{2\gamma} \text{ for the } \left\{\begin{matrix}\text{front end}\\ \text{rear end}\end{matrix}\right\} \text{ of the rod.}$$

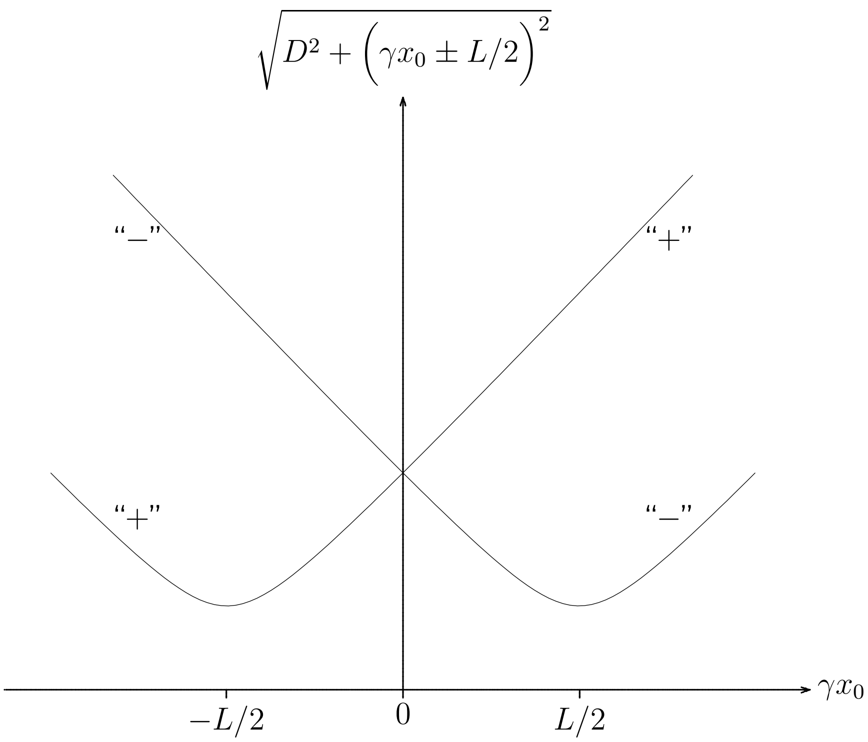
The picture taken by the pinhole camera shows the images of the rod ends at

$$\tilde{x}_\pm = \gamma\left(\gamma x_0 \pm \frac{L}{2}\right) - \beta\gamma\sqrt{D^2 + \left(\gamma x_0 \pm \frac{L}{2}\right)^2}\,.$$

The apparent length $\tilde{L}(x_0) = \tilde{x}_+ - \tilde{x}_-$ is therefore

2.3
$$\tilde{L}(x_0) = \gamma L + \beta\gamma\sqrt{D^2 + \left(\gamma x_0 - \frac{L}{2}\right)^2} - \beta\gamma\sqrt{D^2 + \left(\gamma x_0 + \frac{L}{2}\right)^2}\,.$$

Since the rod moves with the constant speed v, we have $\dfrac{\mathrm{d}x_0}{\mathrm{d}t} = v$ and therefore the question is whether $\tilde{L}(x_0)$ increases or decreases when x_0 increases. We sketch the two square root terms:



The difference of the square roots with "−" and "+" appears in the expression for $\tilde{L}(x_0)$, and this difference clearly decreases when x_0 increases.

2.4	The apparent length decreases all the time.

Symmetric picture For symmetry reasons, the apparent length on the symmetric picture is the actual length of the moving rod, because the light from the two ends was emitted simultaneously to reach the pinhole at the same time, that is:

2.5

$$\tilde{L} = \frac{L}{\gamma}\,.$$

The apparent endpoint positions are such that $\tilde{x}_- = -\tilde{x}_+$, or

$$0 = \tilde{x}_+ + \tilde{x}_- = 2\gamma^2 x_0 - \beta\gamma\sqrt{D^2 + \left(\gamma x_0 + \frac{L}{2}\right)^2} - \beta\gamma\sqrt{D^2 + \left(\gamma x_0 - \frac{L}{2}\right)^2}\,.$$

In conjunction with

$$\frac{L}{\gamma} = \tilde{x}_+ - \tilde{x}_- = \gamma L - \beta\gamma\sqrt{D^2 + \left(\gamma x_0 + \frac{L}{2}\right)^2} + \beta\gamma\sqrt{D^2 + \left(\gamma x_0 - \frac{L}{2}\right)^2}$$

this tells us that

$$\sqrt{D^2 + \left(\gamma x_0 \pm \frac{L}{2}\right)^2} = \frac{2\gamma^2 x_0 \pm (\gamma L - L/\gamma)}{2\beta\gamma} = \frac{\gamma x_0}{\beta} \pm \frac{\beta L}{2}\,.$$

As they should, both the version with the upper signs and the version with the lower signs give the same answer for x_0, namely

2.6

$$x_0 = \beta\sqrt{D^2 + \left(\frac{L}{2\gamma}\right)^2}\,.$$

The image of the middle of the rod on the symmetric picture is, therefore, located at

$$\begin{aligned}\tilde{x}_0 &= \gamma^2 x_0 - \beta\gamma\sqrt{D^2 + (\gamma x_0)^2}\\ &= \beta\gamma\left(\sqrt{(\gamma D)^2 + \left(\frac{L}{2}\right)^2} - \sqrt{(\gamma D)^2 + \left(\frac{\beta L}{2}\right)^2}\right),\end{aligned}$$

which is at a distance $\ell = \tilde{x}_+ - \tilde{x}_0 = \dfrac{L}{2\gamma} - \tilde{x}_0$ from the image of the front end, that is

2.7

$$\ell = \frac{L}{2\gamma} - \beta\gamma\sqrt{(\gamma D)^2 + \left(\frac{L}{2}\right)^2} + \beta\gamma\sqrt{(\gamma D)^2 + \left(\frac{\beta L}{2}\right)^2}$$

or

$$\ell = \frac{L}{2\gamma}\left[1 - \frac{\dfrac{\beta L}{2}}{\sqrt{(\gamma D)^2 + \left(\dfrac{L}{2}\right)^2} + \sqrt{(\gamma D)^2 + \left(\dfrac{\beta L}{2}\right)^2}}\right].$$

Very early and very late pictures At the very early time, we have a very large negative value for x_0, so that the apparent length on the very early picture is

$$\tilde{L}_{\text{early}} = \tilde{L}(x_0 \to -\infty) = (1+\beta)\gamma L = \sqrt{\frac{1+\beta}{1-\beta}}\,L\,.$$

Likewise, at the very late time, we have a very large positive value for x_0, so that the apparent length on the very late picture is

$$\tilde{L}_{\text{late}} = \tilde{L}(x_0 \to \infty) = (1-\beta)\gamma L = \sqrt{\frac{1-\beta}{1+\beta}}\, L\,.$$

It follows that $\tilde{L}_{\text{early}} > \tilde{L}_{\text{late}}$, that is:

2.8 The apparent length is 3 m on the early picture and 1 m on the late picture.

Further, we have

$$\beta = \frac{\tilde{L}_{\text{early}} - \tilde{L}_{\text{late}}}{\tilde{L}_{\text{early}} + \tilde{L}_{\text{late}}}\,,$$

so that $\beta = \dfrac{1}{2}$ and the velocity is

2.9
$$v = \frac{c}{2}\,.$$

It follows that $\gamma = \dfrac{\tilde{L}_{\text{early}} + \tilde{L}_{\text{late}}}{2\sqrt{\tilde{L}_{\text{early}}\tilde{L}_{\text{late}}}} = \dfrac{2}{\sqrt{3}} = 1.1547$. Combined with

2.10
$$L = \sqrt{\tilde{L}_{\text{early}}\tilde{L}_{\text{late}}} = 1.73\,\text{m}\,,$$

this gives the length on the symmetric picture as

2.11
$$\tilde{L} = \frac{2\tilde{L}_{\text{early}}\tilde{L}_{\text{late}}}{\tilde{L}_{\text{early}} + \tilde{L}_{\text{late}}} = 1.50\,\text{m}\,.$$

SOLUTIONS to Theory Question 3

Digital Camera Two factors limit the resolution of the camera as a photographic tool: the diffraction by the aperture and the pixel size. For diffraction, the inherent angular resolution θ_R is the ratio of the wavelength λ of the light and the aperture D of the camera,

$$\theta_R = 1.22\frac{\lambda}{D}\,,$$

where the standard factor of 1.22 reflects the circular shape of the aperture. When taking a picture, the object is generally sufficiently far away from the photographer for the image to form in the focal plane of the camera where the CCD chip should thus be placed. The Rayleigh diffraction criterion then states that two image points can be resolved if they are separated by more than

3.1

$$\Delta x = f\theta_R = 1.22\lambda\, F\natural\,,$$

which gives

$$\Delta x = 1.22\ \mu\text{m}$$

if we choose the largest possible aperture (or smallest value $F\natural = 2$) and assume $\lambda = 500\,\text{nm}$ for the typical wavelength of daylight

The digital resolution is given by the distance ℓ between the center of two neighboring pixels. For our 5 Mpix camera this distance is roughly

$$\ell = \frac{L}{\sqrt{N_p}} = 15.65\,\mu\text{m}\,.$$

Ideally we should match the optical and the digital resolution so that neither aspect is overspecified. Taking the given optical resolution in the expression for the digital resolution, we obtain

3.2

$$N = \left(\frac{L}{\Delta x}\right)^2 \approx 823\ \text{Mpix}\,.$$

Now looking for the unknown optimal aperture, we note that we should have $\ell \geq \Delta x$, that is: $F\natural \leq F_0$ with

$$F_0 = \frac{L}{1.22\lambda\sqrt{N_0}} = 2\sqrt{\frac{N}{N_0}} = 14.34\,.$$

Since this $F\sharp$ value is not available, we choose the nearest value that has a higher optical resolution,

3.3
$$F_0 = 11\,.$$

When looking at a picture at distance z from the eye, the (small) subtended angle between two neighboring dots is $\phi = \ell/z$ where, as above, ℓ is the distance between neighboring dots. Accordingly,

3.4
$$z = \frac{\ell}{\phi} = \frac{2.54 \times 10^{-2}/300\,\text{dpi}}{5.82 \times 10^{-4}\,\text{rad}} = 14.55\,\text{cm} \approx 15\,\text{cm}\,.$$

Hard-boiled egg All of the egg has to reach coagulation temperature. This means that the increase in temperature is

$$\Delta T = T_\text{c} - T_0 = 65\,^\circ\text{C} - 4\,^\circ\text{C} = 61\,^\circ\text{C}\,.$$

Thus the minimum amount of energy that we need to get into the egg such that all of it has coagulated is given by $U = \mu V C \Delta T$ where $V = 4\pi R^3/3$ is the egg volume. We thus find

3.5
$$U = \mu \frac{4\pi R^3}{3} C(T_\text{c} - T_0) = 16768\ \text{J}\,.$$

The simplified equation for heat flow then allows us to calculate how much energy has flown into the egg through the surface per unit time. To get an approximate value for the time we assume that the center of the egg is at the initial temperature $T = 4\,^\circ\text{C}$. The typical length scale is $\Delta r = R$, and the temperature difference associated with it is $\Delta T = T_1 - T_0$ where $T_1 = 100\,^\circ\text{C}$ (boiling water). We thus get

3.6
$$J = \kappa(T_1 - T_0)/R = 2458\,\text{W}\,\text{m}^{-2}\,.$$

Heat is transferred from the boiling water to the egg through the surface of the egg. This gives

3.7
$$P = 4\pi R^2 J = 4\pi\kappa R(T_1 - T_0) \approx 19.3\,\mathrm{W}$$

for the amount of energy transferred to the egg per unit time. From this we get an estimate for the time τ required for the necessary amount of heat to flow into the egg all the way to the center:

3.8
$$\tau = \frac{U}{P} = \frac{\mu C R^2}{3\kappa} \frac{T_\mathrm{c} - T_0}{T_1 - T_0} = \frac{16768}{19.3} = 869\ \mathrm{s} \approx 14.5\,\mathrm{min}\,.$$

Lightning The total charge Q is just the area under the curve of the figure. Because of the triangular shape, we immediately get

3.9
$$Q = \frac{I_0 \tau}{2} = 5\,\mathrm{C}\,.$$

The average current is

3.10
$$I = Q/\tau = \frac{I_0}{2} = 50\,\mathrm{kA}\,,$$

simply half the maximal value.

Since the bottom of the cloud gets negatively charged and the ground positively charged, the situation is essentially that of a giant parallel-plate capacitor. The amount of energy stored just before lightning occurs is $QE_0h/2$ where E_0h is the voltage difference between the bottom of the cloud and the ground, and lightning releases this energy. Altogether we thus get for one lightning the energy $QE_0h/2 = 7.5 \times 10^8$ J. It follows that you could light up the 100 W bulb for the duration

3.11
$$t = \frac{32 \times 10^6}{6.5 \times 10^9} \times \frac{7.5 \times 10^8\,\mathrm{J}}{100\,\mathrm{W}} \approx 10\,\mathrm{h}\,.$$

Capillary Vessels Considering *all* capillaries, one has

$$R_\mathrm{all} = \frac{\Delta p}{D} = 10^7\,\mathrm{Pa\,m^{-3}\,s}\,.$$

All capillaries are assumed to be connected in parallel. The analogy between Poiseuille's and Ohm's laws then gives the hydraulic resistance R of one capillary as

$$\frac{1}{R_{\text{all}}} = \frac{N}{R}\,.$$

We thus get

$$N = \frac{R}{R_{\text{all}}}$$

for the number of capillary vessels in the human body. Now calculate R using Poiseuille's law,

$$R = \frac{8\eta L}{\pi r^4} \approx 4.5 \times 10^{16}\,\text{kg}\,\text{m}^{-4}\,\text{s}^{-1}\,,$$

and arrive at

3.12

$$N \approx \frac{4.5 \times 10^{16}}{10^7} = 4.5 \times 10^9\,.$$

The volume flow is $D = S_{\text{all}}v$ where $S_{\text{all}} = N\pi r^2$ is the *total* cross-sectional area associated with all capillary vessels. We then get

3.13

$$v = \frac{D}{N\pi r^2} = \frac{r^2 \Delta p}{8\eta L} = 0.44\,\text{mm}\,\text{s}^{-1}\,,$$

where the second expression is found by alternatively considering one capillary vessel by itself.

Skyscraper When the slab is at height z above the ground, the air in the slab has pressure $p(z)$ and temperature $T(z)$ and the slab has volume $V(z) = Ah(z)$ where A is the cross-sectional area and $h(z)$ is the thickness of the slab. At any given height z, we combine the ideal gas law

$$pV = NkT \qquad (N \text{ is the number of molecules in the slab})$$

with the adiabatic law

$$pV^\gamma = \text{const} \quad \text{or} \quad (pV)^\gamma \propto p^{\gamma-1}$$

to conclude that $p^{\gamma-1} \propto T^\gamma$. Upon differentiation this gives $(\gamma-1)\dfrac{dp}{p} = \gamma\dfrac{dT}{T}$, so that

3.14

$$\frac{dT}{T} = (1 - 1/\gamma)\frac{dp}{p}\,.$$

Since the slab is not accelerated, the weight must be balanced by the force that results from the difference in pressure at the top and bottom of the slab. Taking downward forces as positive, we have the net force

$$0 = Nmg + A[p(z+h) - p(z)] = \frac{pV}{kT}mg + \frac{V}{h}\frac{dp}{dz}h\,,$$

so that $\dfrac{dp}{dz} = -\dfrac{mg}{k}\dfrac{p}{T}$ or

3.15

$$dp = -\frac{mg}{k}\frac{p}{T}dz\,.$$

Taken together, the two expressions say that

$$dT = -(1 - 1/\gamma)\frac{mg}{k}dz$$

and therefore we have

$$T_{\text{top}} = T_{\text{bot}} - (1 - 1/\gamma)\frac{mgH}{k}$$

for a building of height H, which gives

3.16

$$T_{\text{top}} = 20.6\,^{\circ}\text{C}$$

for $H = 1\,\text{km}$ and $T_{\text{bot}} = 30\,^{\circ}\text{C}$.