Solution E1 /version 3 (Important: In this document decimal comma is used instead of decimal point in graphs and tables)

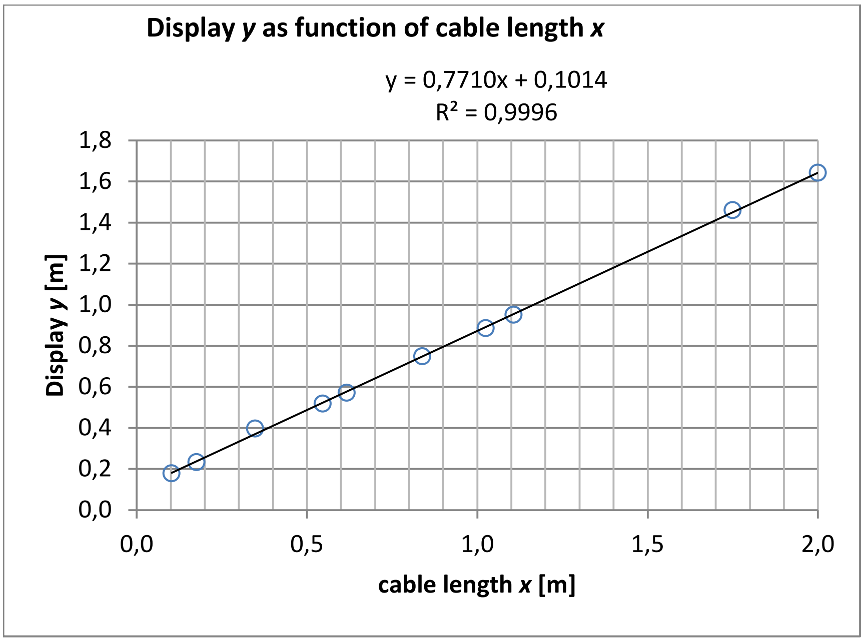
1.1

$H = 907\text{ mm} \pm 2\text{ mm}$. See the sketch in the figure corresponding to 1.3b. It must appear how the height is measured with the LDM in the rear mode.

1.2a

I used a 2 m cable but 1 m is sufficient. There should be about 8 lengths evenly distributed in the interval [0; 1 m].

x	*y*
m	m
0,103	0,177
0,176	0,232
0,348	0,396
0,546	0,517
0,617	0,570
0,839	0,748
1,025	0,885
1,107	0,950
1,750	1,459
2,000	1,642



1.2b

The refractive index is twice the gradient of the linear graph, $n_{\text{co}} = 2 \cdot 0.7710 = 1.542$.

The reason for that is that the travel time for a light pulse

$$t = \frac{x}{v_{\text{co}}} = \frac{x n_{\text{co}}}{c}$$

The display will therefore show $y = \frac{1}{2}ct + k \Leftrightarrow y = \frac{1}{2}n_{\text{co}}x + k$.

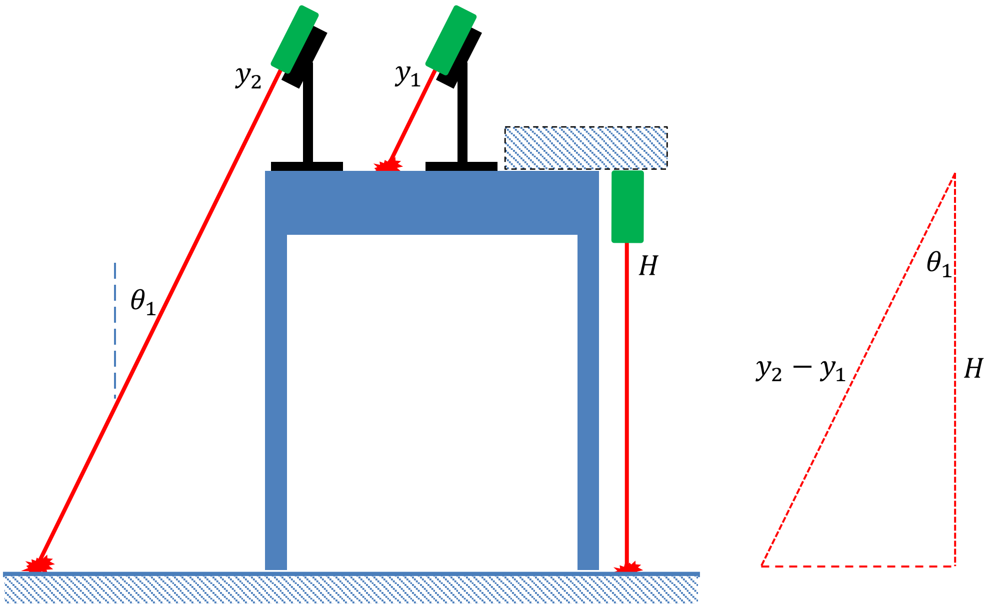
Lysets fart i lyslederkablet er $v_{\text{co}} = \frac{c}{n_{\text{co}}} = \frac{3{,}00 \cdot 10^8 \frac{\text{m}}{\text{s}}}{1{,}542} = 1.95 \cdot 10^8 \,\frac{\text{m}}{\text{s}}$

1.3a

$y_1 = 312\text{ mm} \pm 2\text{ mm},\ y_2 = 1273\text{ mm} \pm 2\text{ mm}$

1.3b

$\theta_1 = \cos^{-1}\left(\frac{H}{y_2 - y_1}\right) = \cos^{-1}\left(\frac{907\text{ mm}}{961\text{ mm}}\right) = 19.30°$, se figure:



Measuring the horizontal part of some triangle is very inaccurate because of the size of the laser dot. No marks will be awarded for that

Using $\delta = 2\text{ mm}$ as the uncertainty of y_1, y_2 and H, one can calculate the uncertainty of θ_1

$$\Delta \cos\theta_1 = \Delta\left(\frac{H}{y_2 - y_1}\right)$$

Using simple derivative, we get

$$\sin\theta_1 \cdot \Delta\theta_1 = \frac{\delta}{H} + \frac{2\delta}{y_2 - y_1}$$

$$\Delta\theta_1 = \frac{\left(\frac{\delta}{H} + \frac{2\delta}{y_2 - y_1}\right)}{\sin\theta_1} \cdot \frac{180°}{\pi} = \frac{\left(\frac{2}{907} + \frac{4}{961}\right)}{\sin 19{,}30°} \cdot \frac{180°}{\pi} = 1.1°$$

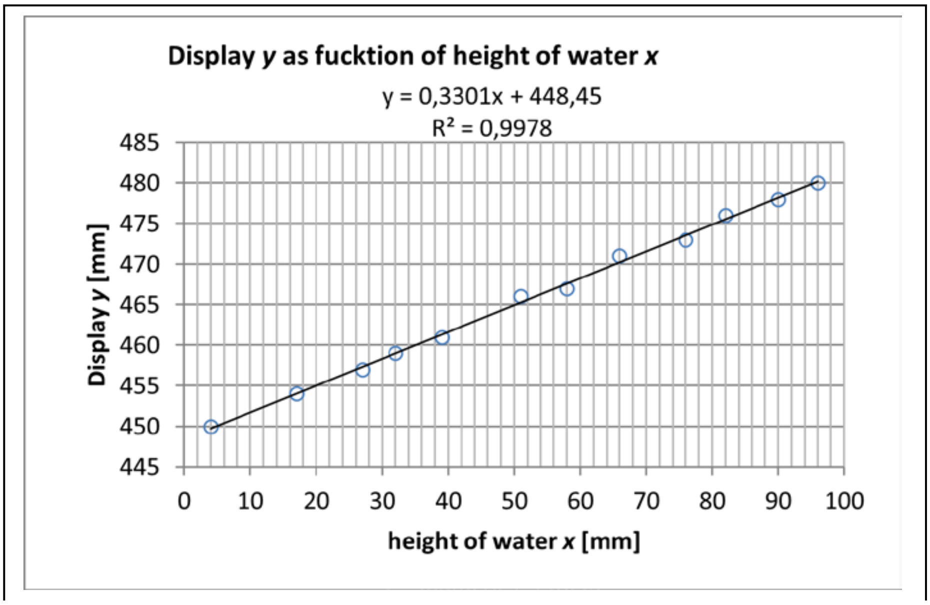
Otherwise, using min/max method

$$\Delta\theta_1 = \theta_{1\max} - \theta_1 = \cos^{-1}\left(\frac{H_{\min}}{y_{2\max} - y_{1\min}}\right) = \cos^{-1}\left(\frac{905\text{ mm}}{965\text{ mm}}\right) - \cos^{-1}\left(\frac{907\text{ mm}}{961\text{ mm}}\right) = 1.0°$$

Also, accept $\delta = 1\text{ mm}$ and $\Delta\theta_1 = 0.5°$

1.4a

x	y
mm	mm
4	450
17	454
27	457
32	459
39	461
51	466
58	467
66	471
76	473
82	476
90	478
96	480



1.4b

The time it takes the light to reach the water surface is

$$t_1 = \frac{(h - x)/\cos\theta_1}{c}$$

From the water surface to the bottom the light uses the time

$$t_2 = \frac{x/\cos\theta_2}{v}$$

Total travel time forth and back

$$t = 2t_1 + 2t_2 = 2\frac{(h - x)/\cos\theta_1}{c} + 2\frac{x/\cos\theta_2}{v} = 2\frac{h - x}{c\cos\theta_1} + 2\frac{nx}{c\cos\theta_2}$$

Hence, the display will show (we simply write $n = n_{\text{w}}$)

$$y = ½ct + k = \left(\frac{n}{\cos\theta_2} - \frac{1}{\cos\theta_1}\right)x + \frac{h}{\cos\theta_1} + k$$

which is a linear function of x.

Using a trigonometric identity and Snell's law,

$$\cos\theta_2 = \sqrt{1 - \sin^2\theta_2} = \sqrt{1 - \frac{\sin^2\theta_1}{n^2}}$$

we get the gradient to be

$$\alpha = \frac{n}{\sqrt{1 - \frac{\sin^2\theta_1}{n^2}}} - \frac{1}{\cos\theta_1} = \frac{n^2}{\sqrt{n^2 - \sin^2\theta_1}} - \frac{1}{\cos\theta_1}$$

1.4c

Knowing the gradient α from the graph, we can find n solving this equation with respect to n.

Introducing a practical parameter,

$$p = \alpha + \frac{1}{\cos\theta_1}$$

our equation becomes

$$p = \frac{n^2}{\sqrt{n^2 - \sin^2\theta_1}}$$

which can be written

$$n^4 - p^2 n^2 + p^2 \sin^2\theta_1 = 0$$

and solved

$$n_\mathrm{w} = \sqrt{\frac{p^2 \pm \sqrt{p^4 - 4p^2\sin^2\theta_1}}{2}} = \frac{\sqrt{2}}{2} p \sqrt{1 \pm \sqrt{1 - \left(\frac{2\sin\theta_1}{p}\right)^2}}$$

From our graph, we get $\alpha = 0.3301$. From there we find $p = 1.37865$ and hence $n_\mathrm{w} = 1.3437$, omitting negative solutions and solutions less than 1.

The official value of n_w for pure water at normal conditions is $n_\mathrm{w} = 1.331$ for the laser wavelength $\lambda = 635\ \mathrm{nm}$.

Just for your interest, we have the following approximations:
For small angles, we have

$$n_\mathrm{w} \approx \frac{\sqrt{2}}{2} p \sqrt{1 + 1 - \frac{1}{2}\left(\frac{2\sin\theta_1}{p}\right)^2} \approx p\sqrt{1 - \left(\frac{\sin\theta_1}{p}\right)^2} \approx p\left(1 - \frac{1}{2}\left(\frac{\sin\theta_1}{p}\right)^2\right)$$

For very small angles, we get

$$n_\mathrm{w} \approx p \approx \alpha + 1$$

It is much simpler but not recommendable to do the experiment with very small $\theta_1 \approx 0$: Reflections in the water surface will ruin the signal from the bottom.

Marking scheme

General remarks applied if nothing mentioned below
Basically valid solutions are awarded at least half marks. For minor errors deduct ¼ of the possible marks, major error or several minor errors deduct ½ of the possible marks. Wrong units or wrong number of significant are minor errors.
Carry-on mistakes are not penalized if they do not change the nature nor the difficulty of the next problem.
Theory subquestions solved using valid methods but different from the one stipulated in the solution are awarded points.

1.1a	Value $H =$ value $\Delta H =$ Include the sketch on a separate sheet **Value of H 0.2, uncertainty 0.1, sketch 0.1** **Slightly imprecise -0.1, forgettning thickness of table -0.2, not using laser at all: no points.** **Measuring from wrong end is ok**	0.4

1.2a	Table: Include a graph of y as a function of x on a separate sheet **Data only in worksheet: no deduction** **Measurements and table: max 1.0** **0.2 for first data point x < 0.3m** **0.2 for first data point 0.3 m < x < 0.5 m** **0.2 for first data point 0.5 m < x < 1m** **0.2 for data point approximately x = 1m (full length of cable)** **0.2 if total number of data points is 5 or more** **Ugly data / imprecise: -0.2** **No units or no mention of physical quantity in heading: -0.1** **Graph: max 0.8** **0.1 for axes with numbers,** **0.5 for correctly plotted data points (0.1 pr point up to max 0.5),** **0.2 for drawing linear fit line**	1.8
1.2b	Value $n_{co} =$ value $v_{co} =$ Include calculations on a separate sheet **Theory for line equation 0.4, numerical calculations for slope 0.4, calculation and result for n 0.2, calculation and result for c 0.2** **Forgetting factor ½: -0.2** **1 significant digit: -0.5** **4 or more significant digits: -0.2**	1.2

1.3a	Value $y_1 \pm \Delta y_1 =$ value $y_2 \pm \Delta y_2 =$ **Value and uncertainty for y1: 0.1, value and uncertainty for y2: 0.1** **Bad uncertainty -0.1 (only once)**	0.2
1.3b	Value $\theta_1 =$ value $\Delta\theta_1 =$ Include calculations on a separate sheet **Value of theta1 0.2, value of uncertainty 0.2** **Claiming uncertainty on y2-y1 is the same as on y2 or y1: -0.1**	0.4

1.4a	Table: Include the graph of y as function of x on a separate sheet **Table: max 1.0** **0.1 pr data point up to 0.8 (8 data points)** **Stating quantities and units 0.2** **Range: if no points below 1cm or no points above 10 cm: -0.2 in each case** **Ugly data / imprecise: -0.2** **Graph: max 0.6** **axes with labels and numbers 0.1** **0.1 for each point up to 0.4** **0.1 for line**	1.6
1.4b	Include equations on a separate sheet **Getting the ideas right (Snell's law for theta2, adding time traveled) 0.4** **Expressing ideas quantitatively (trigonometry and maths) 0.4** **Concluding straight line 0.4**	1.2
1.4c	Value $n_w =$ Include calculations on a separate sheet **Taking points for line and finding slope 0.3** **Substituting Theta2 0.2** **Setting up quartic equation for n 0.4** **Solving equation and select proper value of n 0.3**	1.2

	Total	**8.0**

2.1 The dependence of the solar cell current on the distance to the light source

$$I(r) = \frac{I_a}{1 + \frac{r^2}{a^2}}$$

2.1a	Measure I as a function of r, and set up a table of your measurements.	1.0
2.1b	Determine the values of I_a and a by the use of a suitable graphical method.	1.0

slot #	r	I	$1/I$	r^2
	mm	mA	1/mA	mm^2
3	9.0	5.440	0.184	81
4	14.5	5.290	0.189	210
5	20.0	5.010	0.200	400
6	25.5	4.540	0.220	650
7	31.0	3.840	0.260	961
8	36.5	3.230	0.310	1332
9	42.0	2.730	0.366	1764
10	47.5	2.305	0.434	2256
11	53.0	1.985	0.504	2809
12	58.5	1.730	0.578	3422
13	64.0	1.485	0.673	4096
14	69.5	1.305	0.766	4830
15	75.0	1.140	0.877	5625
16	80.5	1.045	0.957	6480
17	86.0	0.930	1.075	7396
18	91.5	0.840	1.190	8372
19	97.0	0.755	1.325	9409
20	102.5	0.690	1.449	10506

$$I\left(1 + \frac{r^2}{a^2}\right) = I_a$$

$$r^2 = I_a a^2 \cdot \frac{1}{I} - a^2$$

$$a^2 = 1200\ \text{mm}^2 \pm 100\ \text{mm}^2,$$

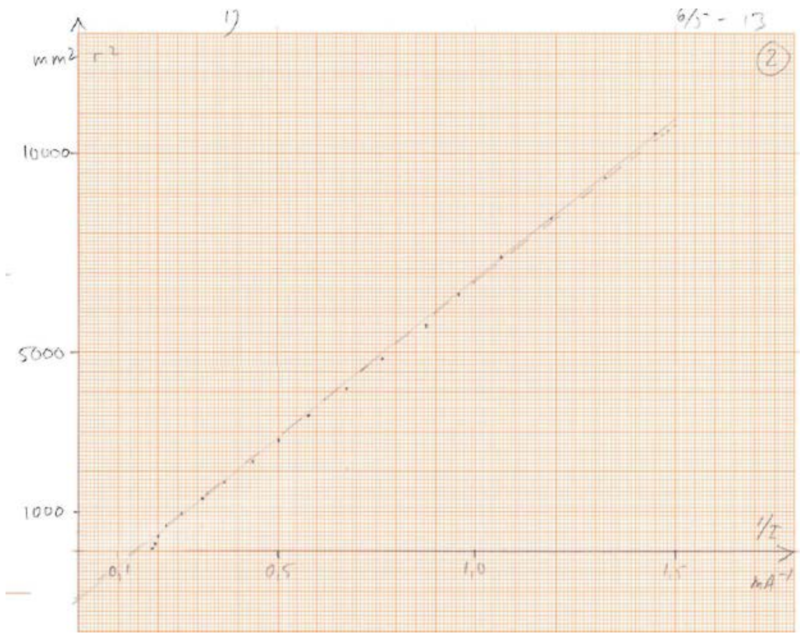
$$a = 35\ \text{mm} \pm \pm 2\ \text{mm}$$

$$I_a a^2 = \frac{10870-0}{1.50-0.15} \cdot \frac{\text{mm}^2}{\text{mA}^{-1}} = 8051.85 \ldots \text{mm}^2\text{mA}$$

$$I_a = \frac{8051.85 \frac{\text{mm}^2}{\text{mA}^{-1}}}{1200\ \text{mm}^2} = 6.7\ \text{mA} \pm 0.5\ \text{mA}$$

$$(I_a a^2)_{\min} = \frac{10700 - 0}{1.50 - 0.14} \cdot \frac{\text{mm}^2}{\text{mA}^{-1}} = 7867.6 \ldots \text{mm}^2\text{mA}$$

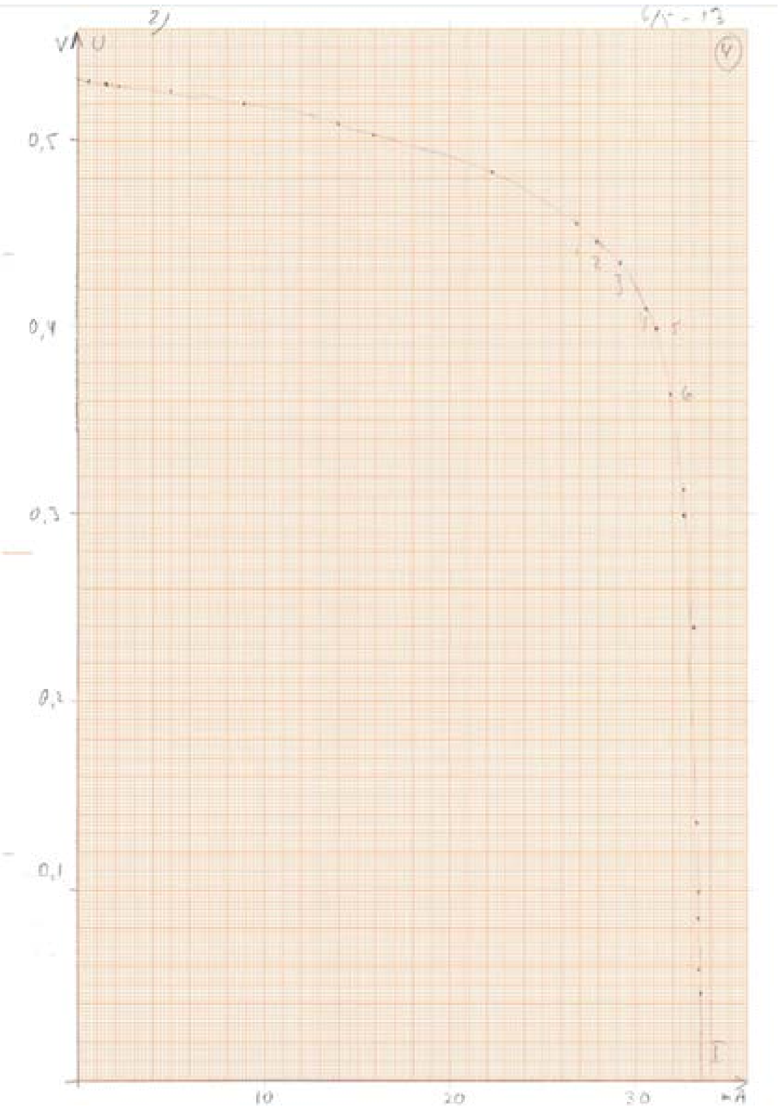
$$\rightarrow I_{a,\max} = \frac{(I_a a^2)_{\min}}{{a^2}_{\min}} = \frac{7867.6\ \text{mm}^2\text{mA}}{1100\ \text{mm}^2} = 7.2\ \text{mA}$$



2.2 Characteristic of the solar cell

2.2a	Make a table of corresponding measurements of *U* and *I*.	0.6
2.2b	Graph voltage as a function of current	0.8

I	*U*
mA	V
0.496	0.532
1.451	0.531
5.05	0.526
8.88	0.52
14.05	0.509
31.1	0.395
25.3	0.471
21.6	0.488
30.6	0.41
31.9	0.364
32.6	0.299
32.6	0.313
33.1	0.239
33.4	0.085
33.3	0.138
33.4	0.096
33.4	0.058
33.5	0.046
33.5	0.045
1.05	0.529
27.8	0.454
15.9	0.503
22.3	0.483
26.8	0.458
29.2	0.435



2.3 Theoretical characteristic for the solar cell

2.3a	Use the graph from question 2.2b to determine $I_{\max}$.	0.4
2.3b	Estimate the range of values of U for which the mentioned approximation is good. Determine graphically the values of I_0 and η for your solar cell.	1.2

$I = I_{\max}$ for $U = 0 \rightarrow I_{\max} = 33.5$ mA

$\eta k_B T < 4 \cdot 1.381 \cdot 10^{-23}\mathrm{J/K} \cdot 300\ \mathrm{K} = 0.103\ \mathrm{eV}$

$$I = I_{\max} - I_0\left(\exp\left(\frac{eU}{\eta k_B T}\right) - 1\right) \approx I_{\max} - I_0 \exp\left(\frac{eU}{\eta k_B T}\right)$$

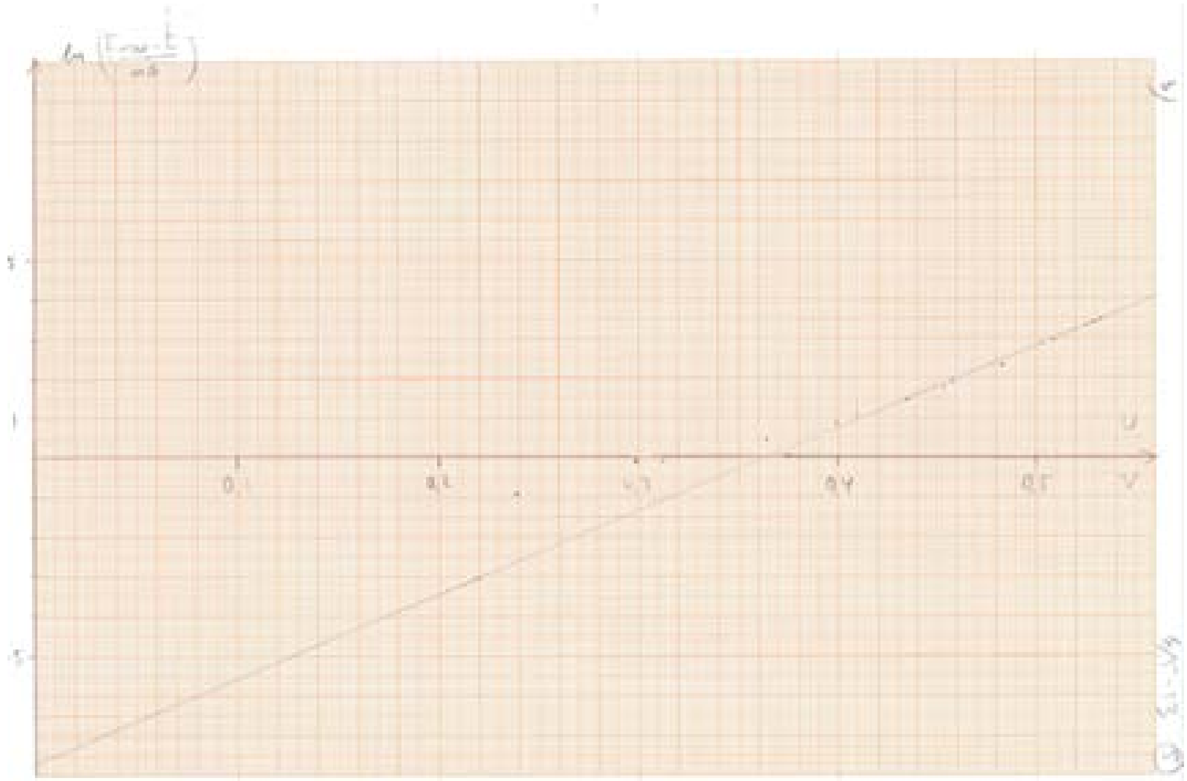
for $U > 0.4\ V$ where $\exp\left(\frac{eU}{\eta k_B T}\right) > \exp(4) \gg 1$

$$ln\left(\frac{I_{max} - I}{\mathrm{mA}}\right) = \frac{e}{\eta k_B T}U - ln\left(\frac{I_0}{\mathrm{mA}}\right)$$

$$\frac{e}{\eta k_B T} = \frac{4.03 - (-7.7)}{0.56\ \mathrm{V}} = 20.95\ \mathrm{V}^{-1}$$

$$I_0 = e^{-7.7}\mathrm{mA} = 0.45\mu A$$

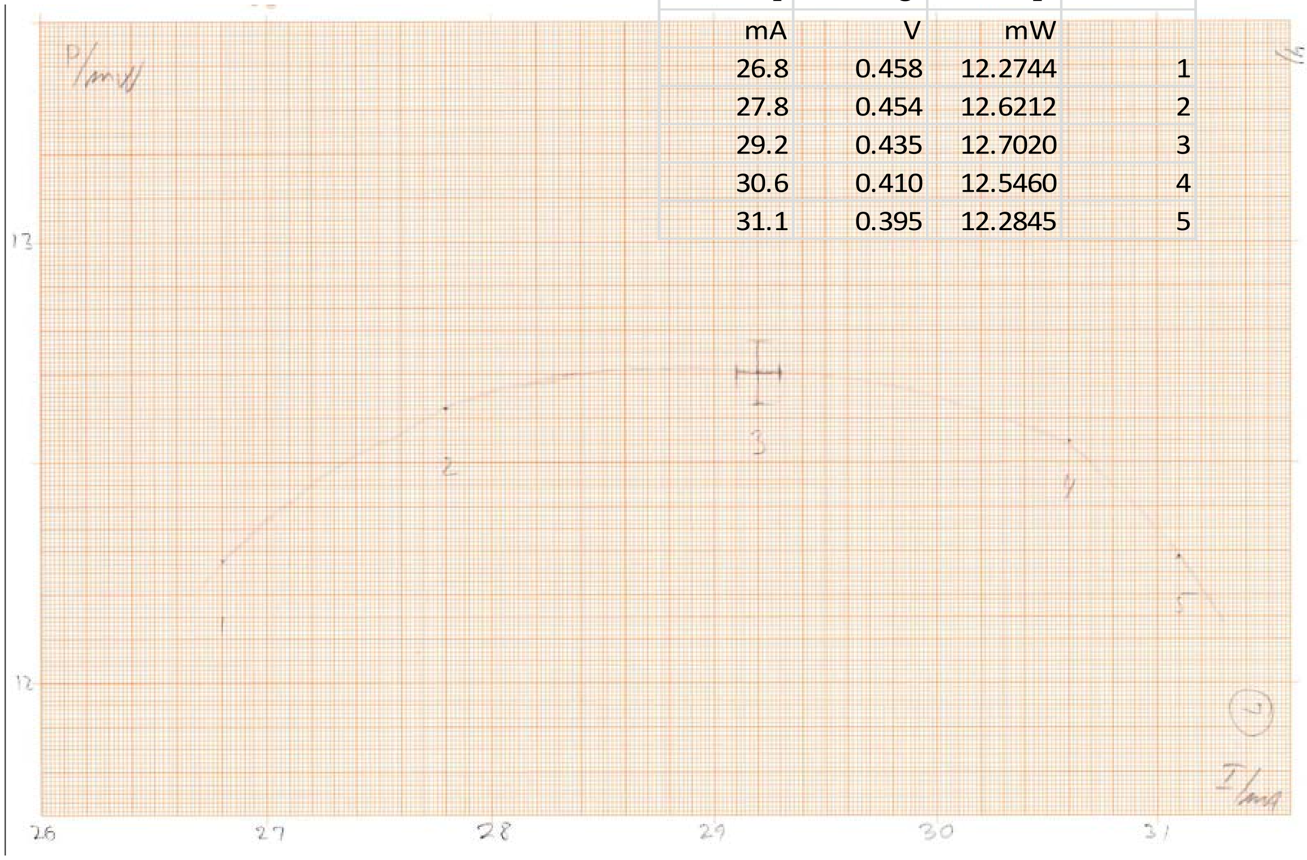
$$\rightarrow \boldsymbol{\eta} = \frac{e/(k_B T)}{20.95\ \mathrm{V}^{-1}} = 1.85$$



2.4 Maximum power for a solar cell

2.4a	The maximum power that the solar cell can deliver to the external circuit is denoted P_{max}. Determine P_{max} for your solar cell through a few, suitable measurements. (You may use some of your previous measurements from question 2.2)	0.5
2.4b	Estimate the optimal load resistance R_{opt}, i.e. the total external resistance when the solar cell delivers its maximum power to R_{opt}. State your result with uncertainty and illustrate your method with suitable calculations.	0.5

I	U	P	
mA	V	mW	
26.8	0.458	12.2744	1
27.8	0.454	12.6212	2
29.2	0.435	12.7020	3
30.6	0.410	12.5460	4
31.1	0.395	12.2845	5



$P_{max} = (12.7 \pm 0.1)\text{mW}@(28.8 \pm 0.2)\text{mA}$

$$R_{opt} = \frac{P_{max}}{I_{opt}^2} = \frac{12.71\text{mW}}{(28.8\text{mA})^2} = (15.3 \pm 0.3)\Omega$$

2.5 Comparing the solar cells

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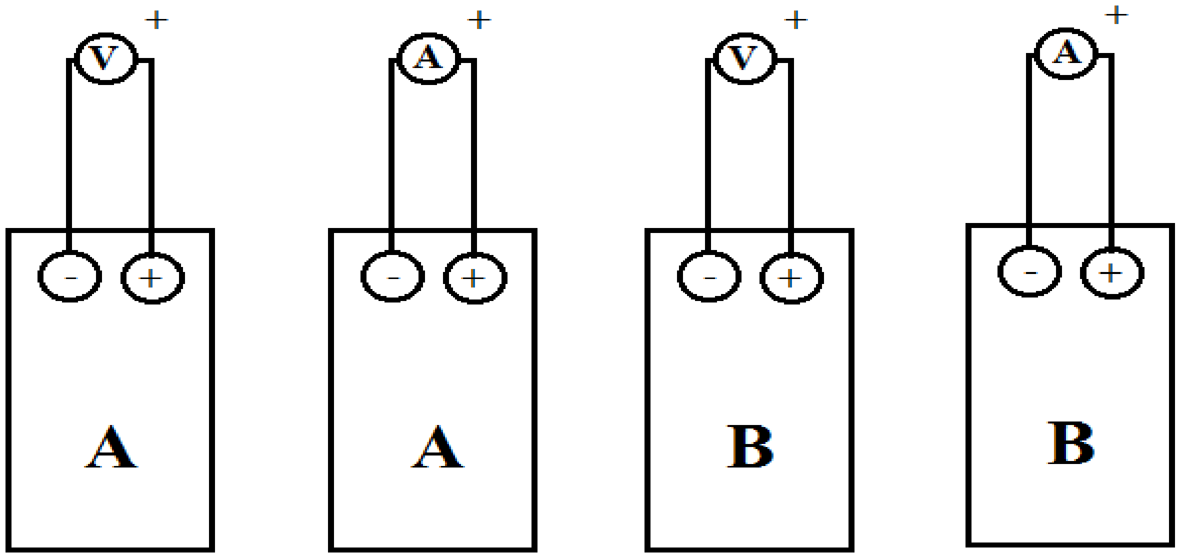
2.5a	Measure, for the given illumination: - The maximum potential difference U_A that can be measured over solar cell A. - The maximum current I_A that can be measured through solar cell A. Do the same for solar cell B.	0.5
2.5b	Draw electrical diagrams for your circuits showing the wiring of the solar cells and the meters.	0.3

2.5a. U_A=0.512 V I_A=16.465 mA U_B=0.480 V I_B = 16.325 mA

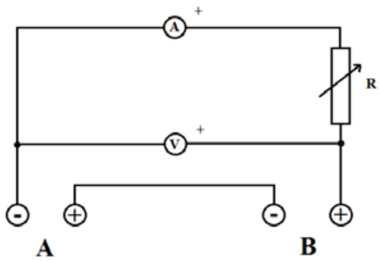
2.5b.



2.6 Couplings of the solar cells

2.6	Determine which of the four arrangements of the two solar cells yields the highest possible power in the external circuit when one of the solar cells is shielded with the shielding plate (J in Fig. 2.1). Draw the corresponding electrical diagram.	1.0

a.



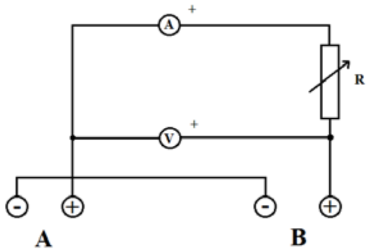
Unshielded (adjusting *R* for reasonalble *P*)

13.10 mA; 0.794 V; 10.4 mW

A shielded: 0.37 mA; 0.022 V

B shielded: 0.83 mA; 0.049V

b.

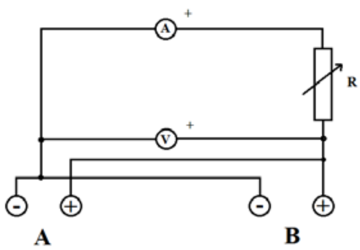


R like in a.

A shielded: 1.47 mA; 0.088V

B shielded: -2.82 mA; -0.170 V

c.

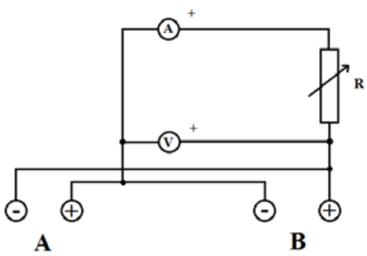


R like in a.

A shielded: 6.89 mA; 0.415 V

B shielded: 6.905 mA; 0.4165 V

d.



R like in a.

A shielded: 7.14 mA; 0.436 V

B shielded: -7.76 mA; -0.474 V

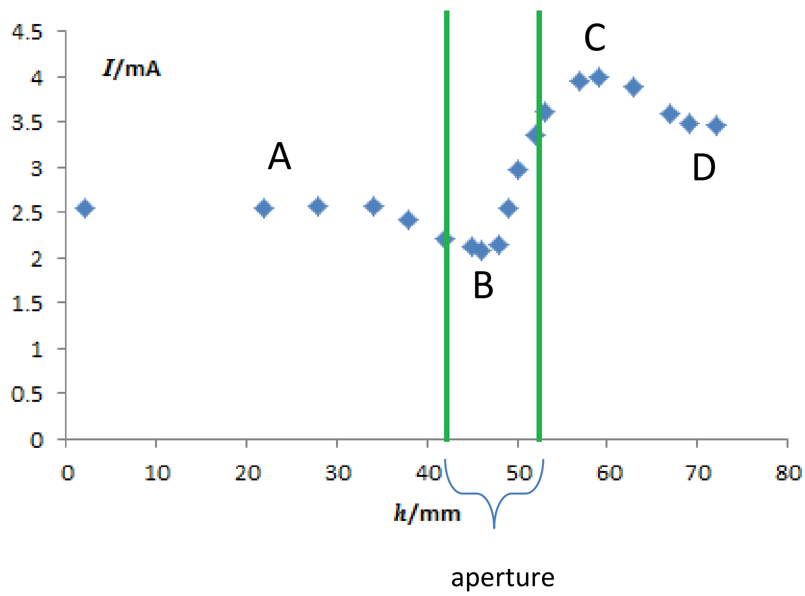
Conclusion: Best power: Set-up d with B shielded. (Solar cell A slightly better than B).

2.7 The effect of the optical vessel (large cuvette) on the solar cell current

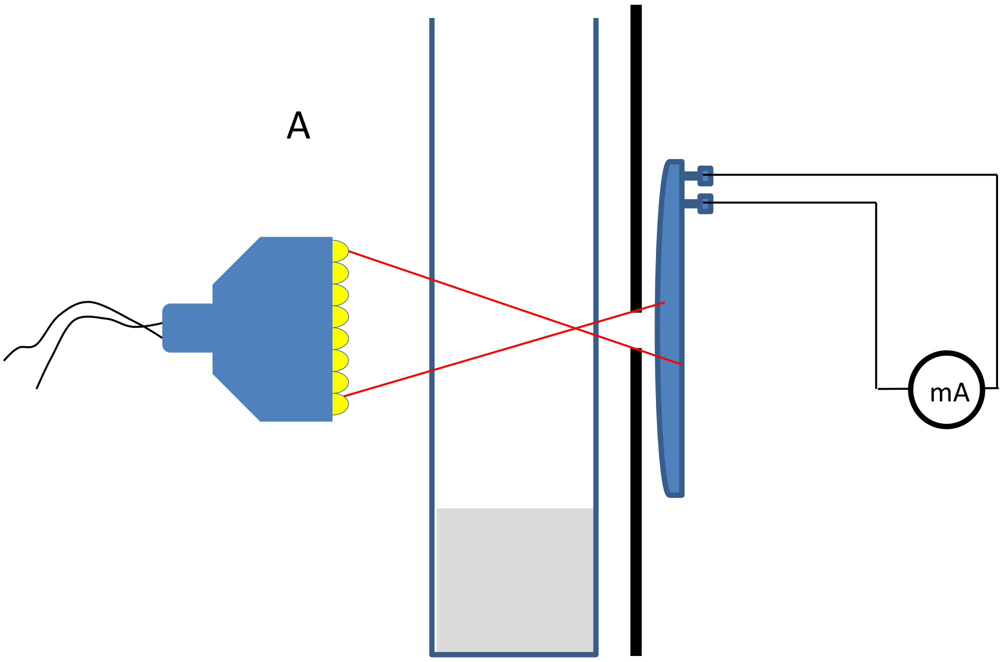
2.7a	Measure the current I, now as a function of the height, h, of water in the vessel, see Fig. 2.8. Make a table of the measurements and draw a graph.	1.0
2.7b	Explain with only sketches and symbols why the graph looks the way it does.	1.0
2.7c	For this set-up do the following: - Measure the distance r_1 between the light source and the solar cell, and the current I_1. - Place the empty vessel immediately in front of the circular aperture and measure the current I_2. - Fill up the vessel with water, almost to the top, and measure the current I_3.	0.6
2.7d	Use your measurements from 2.7c to find a value for the refractive index n_w for water. Illustrate your method with suitable sketches and equations. You may include additional measurements.	1.6

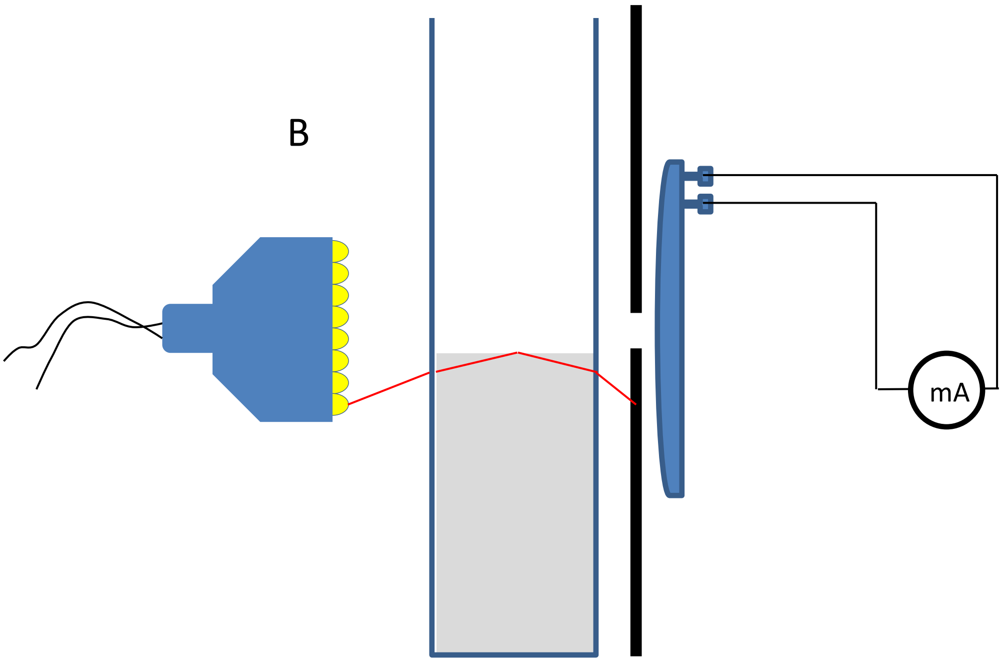
2.7a

h mm	I mA
2	2.54
22	2.55
28	2.56
34	2.57
38	2.42
42	2.21
45	2.13
46	2.08
48	2.15
49	2.54
50	2.97
52	3.36
53	3.61
57	3.96
59	3.99
63	3.89
67	3.6
69	3.49
72	3.47

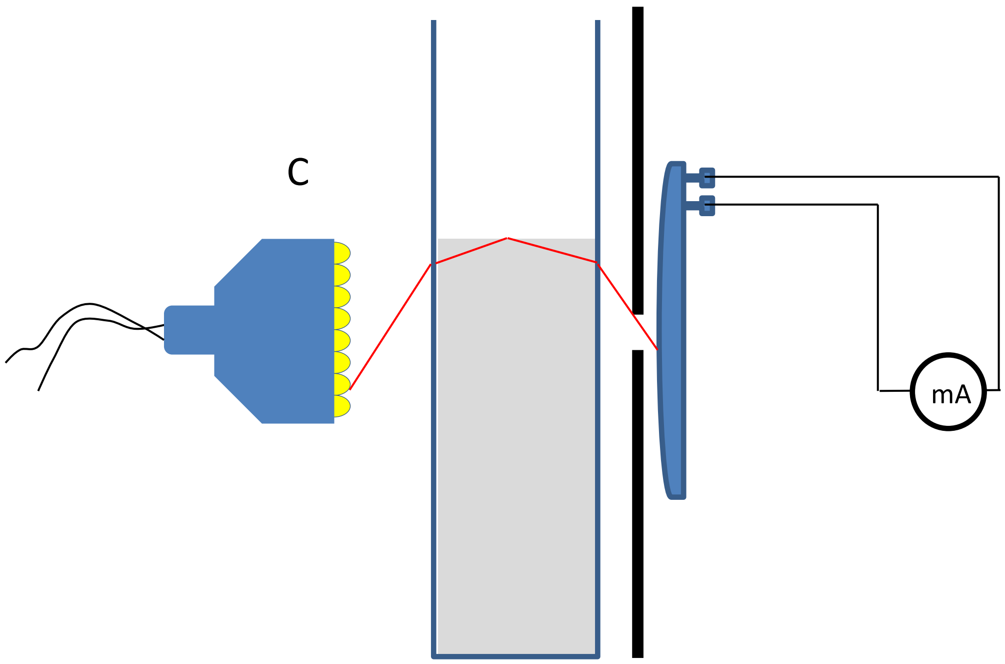


2.7b ***Exemplar*** drawings for position A, B, C and D on previous graph:

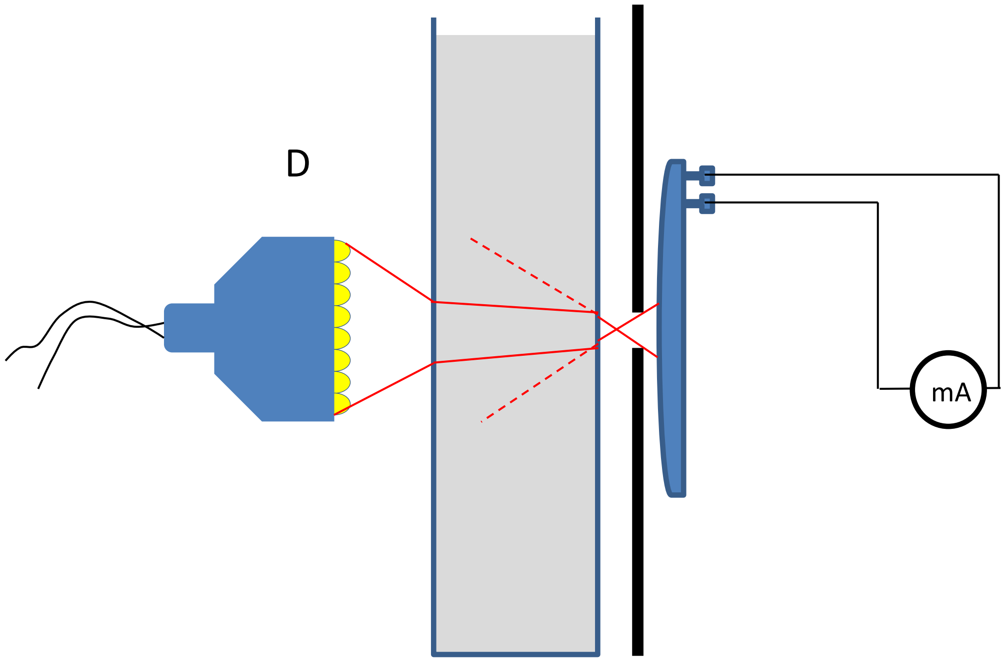




C

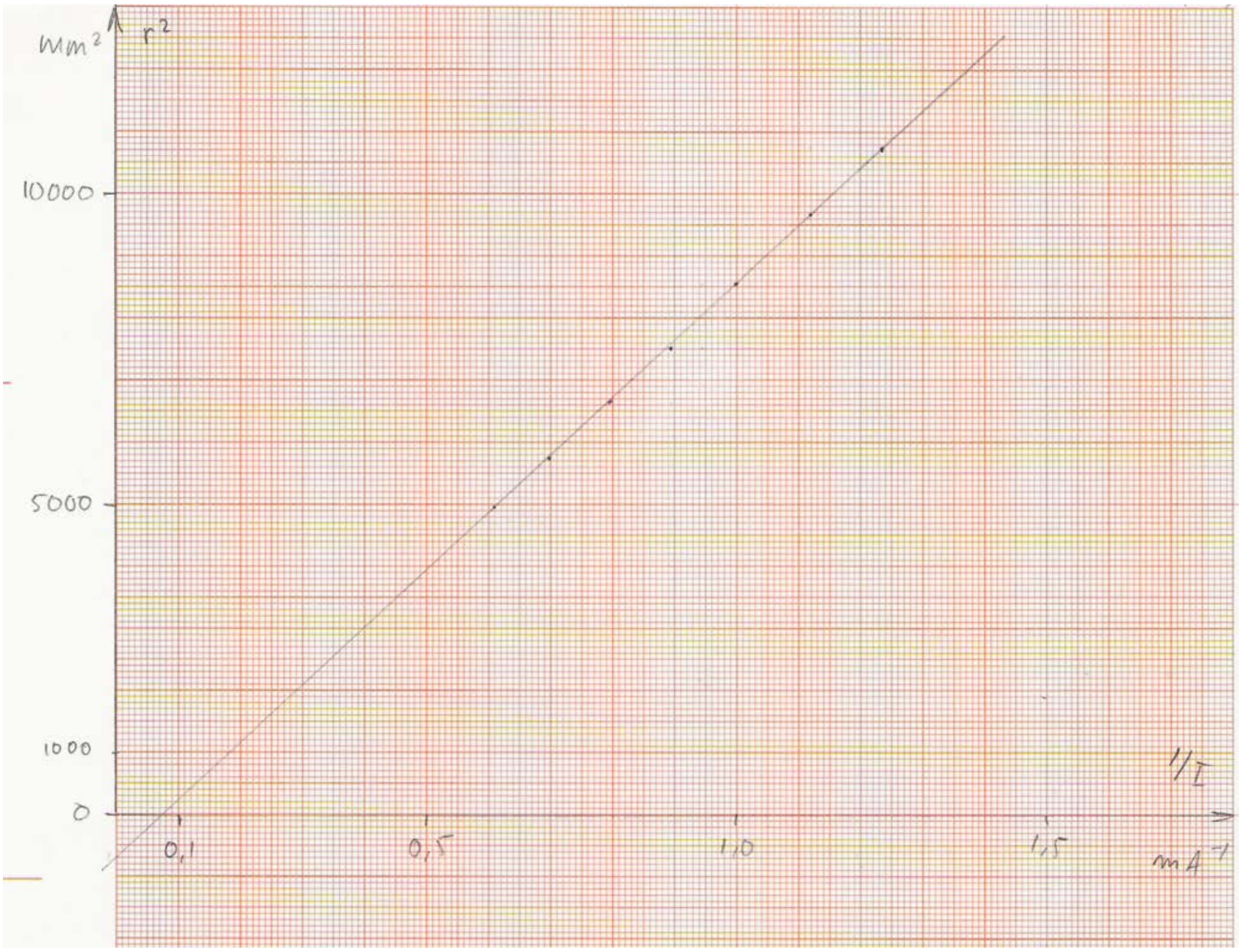


D

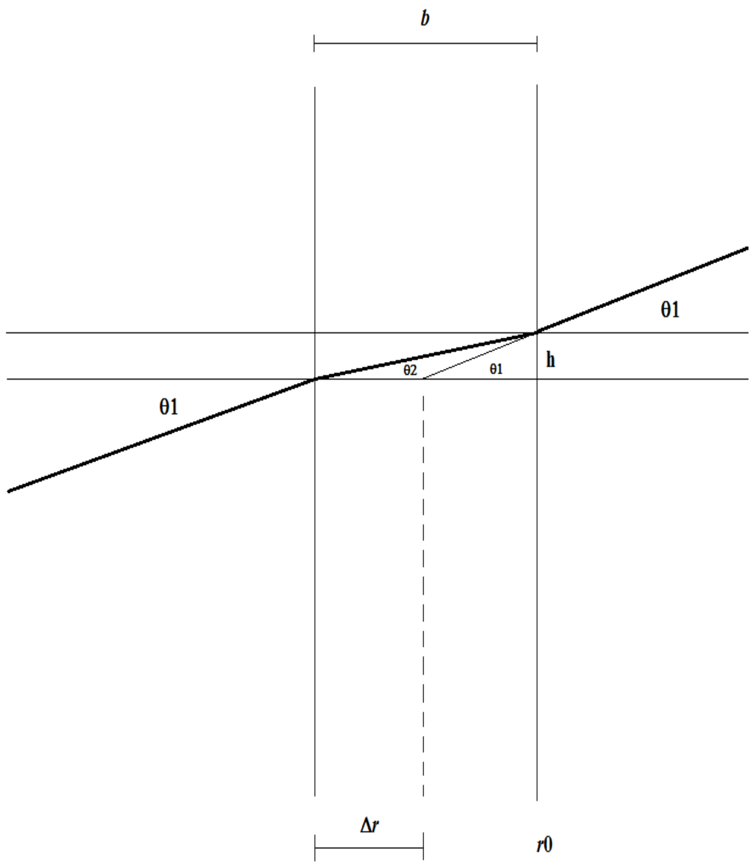


2.7c NOTE: The exemplar measurements are from a different lamp than in 2.1. For a solution to 2.7d using the distance graph you have to refer to the graph below.
$r_1 = 103.5$ mm; $I_1 = 0.81$ mA; $I_2 = 0.705$ mA; $I_3 = 0.85$ mA

$$\frac{1}{I_3} \cdot \frac{I_2}{I_1} = 1.024\ \mathrm{mA}^{-1} \sim r_c^2 = 8800\ \mathrm{mm}^2 \sim r_c = 93.8\ \mathrm{mm}$$



2.7d



$$h = (b - \Delta r)\tan\theta_1 = b\tan\theta_2 \Rightarrow \frac{b}{b-\Delta r} = \frac{\tan\theta_1}{\tan\theta_2} \approx \frac{\sin\theta_1}{\sin\theta_2} = n \text{, da } \theta_2 < \theta_1 << 1.$$

$$n_w \approx \frac{b}{b-\Delta r} = \frac{b}{b-(r_1 - r_c)} = \frac{26.0\text{ mm}}{26.0\text{ mm} - (103.5 - 93.8)\text{mm}} = 1.6$$

NOTE: You may get better results. The uncertainty is rather large in this method because of the subtraction of two large numbers for Δr

A different method is to determine the shift by actually moving the set-up and perhaps making an interpolation in directly measured data.

Marking scheme

General remarks applied if nothing mentioned below

Basically valid solutions are awarded at least half marks. For minor errors deduct ¼ of the possible marks, major error or several minor errors deduct ½ of the possible marks. Wrong units or wrong number of significant are minor errors.

Carry-on mistakes are not penalized if they do not change the nature nor the difficulty of the next problem.

Theory subquestions solved using valid methods but different from the one stipulated in the solution are awarded points.

2.1a	Measurements and table	max	1.0
	0.2 if total number of data points is 8 or more		
	0.2 for first data point < 3 cm		
	0.2 for last data point > 9 cm		
	0.2 for units and symbols		
	0.2 for precise measurements		
2.1b	Linearization		0.3
	Draw graph	max	0.5
	0.1 for axes with numbers and units		
	0.3 correctly plotted data points		
	0.1 for drawing linear fit line		
	Find parameters within 20%	max	0.2
	0.1 pr. parameter		
2.2a	Measurements and table	max	0.6
	0.1 for five points before the "shoulder"		
	0.2 for five points around the "shoulder"		
	0.2 for five points after the "shoulder"		
	0.1 for units and symbols		

2.2b	Draw graph	max	0.8
	0.2 for axes with numbers and units		
	0.3 correctly plotted data points		
	0.3 for a nice and even curve		
2.3a	$U = 0 \Rightarrow I = I_{max}$		0.2
	Extrapolation and reading axis intercept		0.2
2.3b	Estimate the range of U		0.2
	Linearization		0.3
	Draw graph	max	0.5
	0.1 for axes with numbers and units		
	0.2 correctly plotted data points		
	0.2 for a linear fit in the linear range		
	Determination of parameters	max	0.2
	0.1 pr. parameter		
2.4a	Suitable measurements (old or new values)		0.2
	Calculations of P_{max}	max	0.3
	0.1 for calculations of power		
	0.2 for determination of maximum power		
2.4b	Optimal load resistance		0.3
	Uncertainties with correct explanation		0.2
2.5a	Appropriate measurements of four values	max	0.5
	0.2 for first correct measurement		
	0.1 for <u>each</u> of the three subsequent		
	Voltage must be given with three decimal places		
	Current must be given with two decimal places		
	If current >20 mA, only one decimal place is needed		

2.5b	Correct drawings	max	0.3
	0.1 for the first correct voltage diagram		
	0.1 for the first correct current diagram		
	0.1 for second cell		
	Missing indication of polarity will not be penalized		
	OK to state: “similarly for B”		
2.6	Measurements from serial couplings	max	0.3
	0.2 for the first coupling		
	0.1 for the second coupling		
	Measurements from parallel couplings	max	0.5
	0.3 for the first coupling		
	0.2 for the second coupling		
	Drawing of optimal diagram		0.2
2.7a	Measurements and table	max	0.6
	0.1 for four points before the minimum		
	0.1 for five points between minimum and maximum		
	0.1 for four points after the maximum		
	0.1 for point(s) near minimum		
	0.1 for point(s) near maximum		
	0.1 for units and symbols		
	Draw graph	max	0.4
	0.1 for axes with numbers and units		
	0.1 correctly plotted data points		
	0.2 for a nice and even curve		
2.7b	Recognizing the four different situations		0.2
	Drawing of regions	max	0.8
	0.2 for each region		

2.7c The four required values measured max 0.6

0.2 for r and I_1

0.2 for I_2

0.2 for I_3

r within 10 %; and I_2 and I_3 consistent with I_1

2.7d Recognizing the role of refraction max 0.8

0.5 drawing with refractive rays

0.3 use of Schnell's Law

Method 1: Graphical method max 0.8

0.5 for including loss in vessel and scaling argument

0.3 determination of n_w consistent with measurements

Method 2: Measuring values and interpolation max 0.8

0.5 for using interpolation

0.3 determination of n_w consistent with measurements